R N S INSTITUTE OF TECHNOLOGY

CHANNASANDRA, BANGALORE - 61

OPERATIONS RESEARCH

NOTES FOR 6TH SEMESTER INFORMATION SCIENCE
SUBJECT CODE: 06CS661

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REFERENCES:

1. Manjunath Aradhya
2. S D Sharma Text Book
3. Lieberman Text Book
4. Class Notes [Manoj Kumar]

CONTENTS: UNIT 1, UNIT 2, UNIT 3, UNIT 6, UNIT 7

IMPORTANT TIPS

TOPIC-WISE BREAK UP:

1. (a) theory on OR fundamentals
   (b) problem on formulation of a LPP
   (c) problem on graphical method
2. (a) theory on simplex method (including definitions)
   (b) problem on simplex method in tabular form
   (c) problem on trial and error method
3. (a) theory/problem on BIG-M method
   (b) theory/problem on Two-Phase Method.
4. *Revised simplex method – 1 [not covered in the notes]*
5. *Revised simplex method – 2 [not covered in the notes]*
6. (a) problem on transportation models
   (b) problem on assignment models
7. (a) problem on saddle point, dominance rule
   (b) graphical method of solving 2xn or mx2 games/ decision analysis [trees]
8. *Metaheuristics [not covered in the notes]*

PLEASE NOTE:

These are the probable type of questions which we can assume to appear in the VTU examination after the analysis of previous year papers. These may be subjected to changes also.

UNITS 1, 2, 3, 6, 7 are the best and easy ones to prepare and attend in the examination.

Theoretical aspects are almost covered in this notes based on previous paper analysis and we hope it is sufficient in the examination point of view.

Problems such as formulation of LP model require wide subject knowledge since each and every problem is of different variety and there can be no generalisation. Hence, practice more varieties.

Problems based on simplex methods are generally easy to solve once we master the method. But, there are certain conditions such as Degeneracy, tie breaking etc, which have to be taken special care.

Problems on Transportation: important techniques such as VAM, NWC are given more priority whereas RUSSELs method, Travelling salesman, MODI method are dealt with only one example problem since they are not very frequently asked in VTU exam. By analysing an example problem, you must be capable enough to solve similar problems. This applies to assignment problem also. [you can solve anything if you know the steps properly]

Problems on Game theory & Decision analysis: construction of decision trees once again requires good skill and control over the subject since it involves wide varieties of problems.

CLASS ROOM ASSIGNMENTs are attached almost at the end of the notes. PLEASE REFER them since it involves different varieties of problems which are not solved in the notes.

Finally, this NOTES gives a fair idea about the subject and the various methods/techniques. But, since it is a mathematical subject, VTU questions may not be restricted to only the ones solved here..!!!

THE MORE YOU PRACTICE, THE BETTER YOU UNDERSTAND.

ALL THE BEST

Divya

UNIT 1

INTRODUCTION TO LINEAR PROGRAMMING

Origin and Development of Operations Research

Operations Research abbreviated as OR may be described as a scientific approach to decision making. This new subject came into existence during the second world war. OR is defined as an experimental science devoted to observing, understanding & predicting the behaviour of purposeful man-machine systems

Main Phases of OR

The procedure for an OR study involves the following 6 major phases:

Phase 1: Formulation

This phase requires the problem to be formulated in the form of an appropriate model. This includes finding objective function, constraints or restrictions, inter-relationships, possible alternate courses of action, time limits for making decisions, ranges of controllable variables and uncontrollable variables which may affect the possible solutions. A wrong formulation cannot yield a right solution. Hence one must be very careful while executing this phase.

Phase 2: Construction of a Mathematical Model

This phase is concerned with reformation of the problem in an appropriate form which is useful in analysis.

The most suitable model is a mathematical model representing the problem under study. A mathematical model should include decision variables, objective function and constraints. The advantage of a mathematical model is that it describes the problem more concisely which makes the overall structure of the problem more comprehensible and it also helps to reveal important cause and effect relationships.

Phase 3: Derivation of solutions from mathematical model

This phase is devoted to the computation of those values of the decision variables which maximise or minimize (optimise) the objective function. It is always important to arrive at the optimal solution of the problem.

Phase 4: Testing the mathematical model & its solution

The completed model is tested for errors if any. The principle of judging the validity of a model is "whether or not it predicts the relative effects of the alternative ~~causes~~ courses of action with sufficient accuracy to permit a sound decision". A good model should be applicable for a longer time and thus updates the model from time to time by taking into account the past, present and future specifications of the problem.

Phase 5: Establishing Control over the Solution

After the testing phase, the next step is to install a well-documented system for applying the model. It includes the solution procedure and operating procedures for implementation. This phase establishes a control over the solution with some degree of satisfaction. This phase also establishes a systematic procedure for detecting changes and controlling the situation.

Phase 6 : Implementation

The implementation of the controlled solution involves, the translation of the model's results into operating instructions. It is important in OR to ensure that the solution is accurately translated into an operating procedure to rectify faults in the solution.

Advantages of Operations Research

1. Optimum use of Production Factors

Linear Programming techniques indicate how a manager can most effectively employ his production factors by more effectively selecting and distributing these elements.

2. Improved Quality of Decision

The computation table gives a clear picture of the happening within the basic restrictions and the possibilities of compound behaviour of the elements involved in the problem. The effect on profitability due to changes in the production pattern would be clearly indicated in the solution

3. Preparation of Future Managers

OR techniques substitute a means for improving the knowledge and skill of young managers.

4. Modifications of a mathematical solution:

OR presents a possible practical solution when one exists, but it is very important to accept or modify it before using. The effect of these modifications may be evaluated from the computational steps and tables.

5. Alternate solutions

OR techniques will suggest all the alternate solutions available for the same problem so that best solution can be selected.

Disadvantages of OR

1. Practical application

Formulation of an industrial problem as a LP problem is a difficult task.

2. Reliability

A non-linear relationship is changed to linear relationship for fitting the problem to linear programming pattern. This may disturb the system.

3. Magnitude of Computation

OR tries to find out the optimum solution by taking all the factors into account. In practical problems, these factors are numerous and expressing

4. Absence of Quantification

OR provides solution only when all the elements related to a problem are quantified. The tangible factors such as price, product etc. can be expressed in terms of quality. But intangible elements of the system can't be quantified (like human relationship)

5. Distance between managers and OR

OR problems require a mathematician or statistician who might not be aware of the business problems. Similarly, a manager may not understand the complex working mathematical models considered in OR.

Nature of Operations Research

OR involves "research on operations". Thus, operations Research is applied to problems that concern how to conduct and co-ordinate the operations within an organisation. The nature of organisation is immaterial, and, in fact, OR has been applied extensively in such diverse areas as manufacturing, transportation, constructions, telecommunications, financial planning, health care, to name just a few. Therefore, the breadth of application is usually wide.

OR uses an approach that resembles the way research is conducted in established scientific fields. To a considerable extent, the scientific method is used to investigate the problem of concern. OR involves creative scientific research into the fundamental properties of operations.

OR frequently attempts to find a best solution for the problem. Rather than simply improving the status quo, the goal is to identify a best possible course of action.

The impact of Operation Research

Operations Research has had an impressive impact on improving the efficiency of numerous organisations around the world. In the process, OR has made a significant contribution to increasing the productivity of the economics of various countries. There are now a few dozen member countries on the International Federation of OR societies (IFORS), with each country having a national OR society.

Linear Programming:

It deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equations and/or inequalities known as constraints.

Requirements for a LPP

A LPP can be optimized if the following conditions are satisfied:

1) There must be a well defined objective function which can be optimized and can be expressed as a linear function of decision variables.
2) There must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or linear inequalities in terms of decision variables
3) There must be an alternate course of action
4) The decision variables should be inter-related and non-negative.

Some important definitions used in LP-models

1) Solution :

A set of variables $\{x_1, x_2, x_3 \ldots\ldots x_n\}$ is called a solution if it satisfies the constraints.

2) Feasible Solution :

A set of variables $\{x_1, x_2, x_3 \ldots\ldots x_n\}$ is called a feasible solution if these variables satisfy constraints are non -ve.

3) Basic solution:

A solution obtained by setting 'n' variables (among $m+n$ variables) to zero and solving for remaining m variables is called a basic solution. These m variables are basic variables and n variables are non-basic variables.

Total no of basic solutions $= (m+n) C_n$

4) Basic Feasible Solution (BFS)

A basic solution is called a basic feasible solution if all basic variables are ≥ 0.

5) Non - degenerate BFS

It is a BFS in which all m variables are +ve and the remaining 'n' variables are zero each.

6) Degenerate BFS

It is a BFS in which one or more of the m basic variables are equal to zero.

7) Optimal BFS

A BFS is called the optimal BFS if it optimizes the objective function.

8) Unbounded Solution

If the value of the objective function can be increased or decreased indefinitely, then the solution is called an unbounded solution.

9) Feasible Region:

It is a region in which all constraints and non-negativity conditions hold good.

10) Corner Point Feasible (CPF) solution

It is a feasible solution that doesn't lie on any line segment connecting two other feasible solution.

Optimization

It is the technique of obtaining the best results under the given conditions

Programming

It is the mathematical technique to determine the optimum use of the limited available resources.

Linear Programming

It is a decision making technique under the given constraints under the condition that the relationship among the variables involved is linear.

MATHEMATICAL FORMULATION OF A LPP

Step 1:

Define the decision variables x_1, x_2, x_3 ---. etc

Step 2:

Construct the objective function which has to be optimized as a linear equation involving decision variables.

Step 3:

Express every condition as a linear inequality involving decision variables.

Step 4:

State the non-negativity condition and hence express the given problem as a mathematical model.

- A manufacturer produces 3 models I, II and III of a certain product using raw materials A and B. The following table gives the data

Raw material	Requirement per Unit			Availability
	I	II	III	
A	2	3	5	4000
B	4	2	7	6000
Min Dem	200	200	150	–
Profit/Unit	30	20	50	–

Formulate this problem as a LPP [JUNE-JULY 2009]

Solution:

Let the total no. of units of model I be 'x'

Let the total no. of units of model II be 'y'

Let the total no. of units of model III be 'z'

Given profit/one unit of model I is Rs. 30

profit/x unit of model I is $30x$

profit/one unit of model II is Rs. 20

profit/y unit of model II is $20y$

profit/one unit of model III is Rs. 50

profit/z unit of model III is $50z$

Total Profit $= P = 30x + 20y + 50z$

$\therefore$ Objective function is Max $P = 30x + 20y + 50z$

Given that models I, II and III require 2, 3 and 5 units of raw materials A respectively.

$\Rightarrow$ total raw-material A requirement $= 2x + 3y + 5z$

But total availability of raw-material is 4000.

$\Longrightarrow 2x + 3y + 5z \leq 4000$

Given that models I, II and III require 4, 2 and 7 units of raw material B respectively.

$\Rightarrow$ total raw material B requirement $= 4x + 2y + 7z$

But total availability units of raw material $B = 6000$

$\Longrightarrow 4x + 2y + 7z \leq 6000$

Given: Min demand for model I is 200 $\Rightarrow x \geq 200$

Min demand for model II is 200 $\rightarrow y \geq 200$

Min demand for model III is 150 $\Rightarrow z \geq 150$

The total no. of units of model I, model II and model III cannot be negative i.e. $x \geq 0, y \geq 0, z \geq 0$

LP model:

$$\text{Max } P = 30x + 20y + 50z$$

$$\begin{aligned} \text{STC} \quad 2x + 3y + 5z &\leq 4000 \\ 4x + 2y + 7z &\leq 6000 \\ x &\geq 200 \\ y &\geq 200 \\ z &\geq 150 \end{aligned}$$

where $x \geq 0, y \geq 0, z \geq 0$

- A TV company has to decide on the number of 27-inch and 20-inch TV sets to be produced at one of its factories. Market research indicates that atmost 40 27-inch TV sets and atmost 10 20-inch TV sets can be sold per month. The maximum no of work hours available is 500 hours/ month. A 27-inch TV requires 20 workhours and a 20-inch TV requires 10 work-hours. Each 27-inch TV sold produces a profit of $120~~0~~ and each 20-inch TV produces a $80. A wholesaler agreed to purchase all the TV sets produced if the no do not exceed the maximum indicated by market-research. Formulate this problem as a LP model. [DEC 2009]

Solution :

Let the total number of 27-inch TVs be 'x'

Let the total number of 20-inch TV's be 'y'.

Given 1 unit of 27-inch TV produces a profit of $120

→ x units of 27-inch TV produces a profit of $120x$

Given 1 unit of 20-inch TV produces a profit of $80

→ y units of 20-inch TV produces a profit of $80y$

Total Profit $= 120x + 80y$

Objective Function : Max $P = 120x + 80y$

Given that Max sales of 27-inch TV is 40 $\Rightarrow$ $x \leq 40$

Given that Max Sales of 20-inch TV is 10 $\Rightarrow$ $y \leq 10$

Given that one 27-inch TV requires 20 work hours.

→ x 27-inch TV requires $20x$ workhours

Given that one 20-inch TV requires 10 work hours

→ y 20-inch TV require $10y$ workhours.

$\therefore$ Total work hours required $= 20x + 10y$

But total workhours available is 500

$$\rightarrow 20x + 10y \leq 500$$

Max sales/month $= 40 + 10 = 50$

Total no of TV sets $= x + y$

Given: wholesaler will purchase all TV sets if the total doesnot exceed the maximum.

$$\Rightarrow x + y \leq 50$$

We note that total no of 27-inch TVs manufactured cannot be negative i.e. $x \geq 0$.

We note that total no of 20-inch TVs manufactured cannot be negative i.e. $y \geq 0$.

LP Model

Maximize $P = 120x + 80y$

STC

$$x \leq 40$$

$$y \leq 10$$

$$20x + 10y \leq 500$$

$$x + y \leq 50$$

where $x \geq 0,\ y \geq 0$

- A farmer has to plant two kinds of trees P and Q in a land of 4000 sq-mt area. Each tree P requires 25 sq.m. Each tree Q requires 40 sq.m of land. The annual water requirement for each tree P is 30 units and for each tree Q is 15 units. Atmost 3000 units of water is available annually. It is also estimated that the ratio of the total no of trees Q to the no of trees P should not be less than 6/19 and should not be more than 17/8. The return per tree from P is expected to be one ~~mo~~ and a half times as much as from tree Q. Formulate the problem as a LP model.
[JUNE 2010]

Solution :

Profit /one tree Q is 1 $\Rightarrow$ profit wrt tree Q is y

profit / one tree P is 1.5 $\Rightarrow$ Profit wrt tree P is $1.5x$

Total profit $= 1.5x + y$

Objective Function : Max $P = 1.5x + y$

Given that one-tree p requires 25 sq.m

Then, x tree P require $25x$

Given that one-tree Q requires 40 sq.m

Then, y tree Q require $40y$

Total land requirement $= 25x + 40y$

But total land available is 4000

$\Rightarrow 25x + 40y \leq 4000$

Given that one tree P requires 30 units of water

$\rightarrow$ x-tree P require $30x$ units of water

Given that one tree Q requires 15 units of water

$\rightarrow$ y-tree Q require $15y$ units of water

∴ Total water requirement = $30x + 15y$

But atmost 3000 units of water are available

$\Rightarrow 30x + 15y \leq 3000$

Given $\dfrac{\text{total trees-Q}}{\text{total trees-P}}$ should not be less than $\dfrac{6}{19}$

$$\Rightarrow \frac{y}{x} \geq \frac{6}{19}$$

Given $\dfrac{\text{total trees-Q}}{\text{total trees-P}}$ should not be more than $\dfrac{17}{8}$

$$\Rightarrow \frac{y}{x} \leq \frac{17}{8}$$

Total no. of trees P cannot be negative i.e. $x \geq 0$

Total no. of trees Q cannot be negative i.e. $y \geq 0$

LP Model

Maximize $P = 1.5x + y$

STC $30x + 15y \leq 3000$

$25x + 40y \leq 4000$

$$\frac{y}{x} \geq \frac{6}{19}$$

$$\frac{y}{x} \leq \frac{17}{8}$$

$x \geq 0, \quad y \geq 0$

- A company manufactures three products P_1, P_2 and P_3. The profits are 30, 20 and 40. Company has two machines M_1 and M_2. Processing time in minutes for each machine on each product an the total time availability on each machine are given in the following table

Machine	Machine minutes required			Total Time Availability
	P_1	P_2	P_3	
M_1	4	3	5	2000
M_2	2	2	4	5000

Company must manufacture atleast 100 P_1's and atleast 200 P_2's and 50 P_3's but not more than 150 P_1's. Set up LP model for the given problem to solve by Simplex method.
[JUNE 2011]

Solution :

Let the total no of units of P_1 be x

Let the total no of units of P_2 be y

Let the total no of units of P_3 be z

Given profit/unit of $P_1 = 30 \implies$ Profit/x units of $P_1 = 30x$

Given profit/unit of $P_2 = 20 \implies$ Profit/y units of $P_2 = 20y$

Given profit/unit of $P_3 = 40 \implies$ Profit/z units of $P_3 = 40z$

$$\text{Total Profit} = 30x + 20y + 40z$$

Objective Function: Maximize $P = 30x + 20y + 40z$

Given P_1, P_2, P_3 require 4, 3 and 5 minutes of processing on M_1.

Total ~~no.~~ ~~of~~ minutes of processing on M_1

$$= 4x + 3y + 5z$$

But M_1 is available for a max of 2000 minutes

$$\Rightarrow 4x + 3y + 5z \leq 2000$$

Given P_1, P_2 and P_3 require 2, 2 and 4 minutes of processing on M_2.

Total minutes of processing on $M_2 = 2x + 2y + 4x$

But M_2 is available for max for 5000 minutes

$$\Rightarrow 2x + 2y + 4z \leq 5000$$

It is also said that company manufactures atleast 100 P_1 s

——————〃—————— 200 P_2 s

——————〃—————— 50 P_3 s

——————〃—————— not more than 150 P_1 s

i.e.

$$\left.\begin{array}{l} x \geqslant 100 \\ x \leq 150 \end{array}\right\}$$

$$y \geqslant 200$$

$$z \geqslant 50$$

The total number of products P_1, P_2, P_3 manufactures cannot be negative i.e. $x \geqslant 0, y \geqslant 0, z \geqslant 0$.

LP model

Maximize $Z = 30x + 20y + 40z$

STC

$$4x + 3y + 5z \leq 2000$$

$$2x + 2y + 4z \leq 5000$$

$$\left.\begin{array}{l} x \geqslant 100 \\ x \leq 150 \end{array}\right\} \text{(or)} \quad 100 \leq x \leq 150$$

$$y \geqslant 200$$

$$z \geqslant 50$$

$$x \geqslant 0, y \geqslant 0, z \geqslant 0$$

- A farmer has 100 acre farm. He call sell all tomatoes, lettuce or radishes he can raise. The price he can obtain is ₹ 1.00 per kg of tomatoes, ₹ 0.75 a head for lettuce and ₹ 2.00 per kg for radishes. The average yield per acre is 2000kgs of tomatoes, 3000 heads of lettuce and 1000kgs of radishes. Fertilizers is available at ₹ 0.50 per kg and the amount required per acre is 100kg each for tomatoes and lettuce and 50kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6-mandays for lettuce. A total of 400 man-days of labour are available at ₹ 20.0 per man-day. Formulate this problem as a linear programming model to maximize the farmer's total profit. [DECEMBER 2011]

Solution

Let the total no of acres of land devoted to tomatoes = x

———————— " ———————— pumpkins = y

———————— " ———————— radishes = z

We know: Profit = Income − Expenditure

Objective Function:

Income:- Given that price of 1kgs of tomatoes = 1

Given that price of 2000kg of tomatoes = 2000

$\Rightarrow$ Amount of money farmer gains = $2000x$

Given that price per one unit of pumpkin = 0.75

Price/3000 units of pumpkin = $3000 \times 0.75 = 2250$

$\Rightarrow$ Income from pumpkins = $2250y$

Given that price per one kg of radish $= 2$

Price/1000 kg of radish $= 2000$

$\Rightarrow$ Income from radish $= 2000x$

$\therefore$ Total income $= 2000x + 2250y + 2000z$

Expenditure: Rate of 1kg of fertilizers $= 0.50$

Given that 100kg of fertilizers is required for tomatoes/acre

$\Rightarrow$ total cost of fertilizers for tomatoes/acre $= 100 \times 0.50 = 50$

III^ly Given that 100kg of fertilizers is required for pumpkins/acre

$\Rightarrow$ total cost of fertilizers for pumpkins/acre $= 100 \times 0.50 = 50$

Also, Given that 50kg of fertilizers is required for radishes/acre

$\Rightarrow$ total cost of fertilizers for radishes/acre $= 50 \times 0.50 = 25$

$\therefore$ Total expenditure towards fertilizers $= 50x + 50y + 25x$

Given that

5 man-days @ ₹ 20 is required for tomatoes/acre

6 man-days @ ₹ 20 is required for pumpkins/acre

5 man-days @ ₹ 20 is required for radishes/acre

For 1acre, labour cost of tomatoes $= 100$

—— " —— pumpkins $= 120$

—— " —— radishes $= 100$

$\therefore$ Total expenditure towards labour $= 100x + 120y + 100z$

$\therefore$ Profit = Income − Expenditure

$$\therefore \text{Profit} = (2000x + 2250y + 2000z) - (50x + 50y + 25z) - (100x + 120y + 100z)$$

Objective Function $P = 1850x + 2080y + 1875z$

Total labour man-day required $= 5x + 6y + 5z$

But total no. of days available $= 400$

$$\Rightarrow 5x + 6y + 5z \leq 400$$

Total land available $= 100$ acres

$$\Rightarrow x + y + z \leq 100$$

It is obvious that total number of acres of farm cannot be negative. $\therefore x \geq 0, y \geq 0, z \geq 0$

LP Model

Maximize $P = 1850x + 2080y + 1875z$

STC $x + y + z \leq 100$

$5x + 6y + 5z \leq 400$

$x, y, z \geq 0$

- Old hens can be bought at ₹2 each and young ones at ₹5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs ₹1 per week to feed. You have only ₹80 to spend for buying hens. How many of each kind should you buy to give a profit of more than ₹6 per week assuming that you cannot house more than 20 hens. Formulate the problem as an LPP and solve graphically. [DECEMBER 2011]

Solution

FORMULATION

Let x_1 be the number of old hens and

x_2 be the number of young hens to be bought

Since old hens ~~can be bought~~ lay 3 eggs per week and the young ones lay 5 eggs per week, the total number of eggs obtained per week will be $= 3x_1 + 5x_2$

Consequently, the cost of each egg being 30 paise, the total gain will be $=$ Rs. $0.30(3x_1 + 5x_2)$

Total expenditure for feeding $(x_1 + x_2)$ hens at the rate of Re. 1 each will be $=$ Rs. $1(x_1 + x_2)$

Thus, total profit z earned per week will be

z = total gain − total expenditure

or $z = 0.30(3x_1 + 5x_2) - (x_1 + x_2)$

or $z = 0.50x_2 - 0.10x_1$

Since old hens can be brought at Rs. 2 each and young ones at Rs. 5 each and there are only Rs. 80

available for purchasing hens, the constraint is :

$$2x_1 + 5x_2 \leq 80$$

Also, since it is not possible to house more than 20 hens at a time, $x_1 + x_2 \leq 20$.

Also, since the profit is restricted to be more than Rs. 6, this means that the profit function z is to be maximized. Thus, there is no need to add one more constraint ie. $0.5x_2 - 0.1x_1 \geq 6$

Again, it is not possible to purchase negative quantity of hens, therefore $x_1 \geq 0, x_2 \geq 0$.

LP Model

Maximize $Z = 0.5x_2 - 0.1x_1$

STC $2x_1 + 5x_2 \leq 80$

$x_1 + x_2 \leq 20$

$x_1, x_2 \geq 0$

NOTE : SOLVE GRAPHICALLY ALSO

- A company produces two types of Hats. Each hat of the first type requires twice as much labour time as the second type. If all the hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type A and Rs. 5 for type B. formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

SOLUTION:

Let the company produce x_1 hats of type A and x_2 hats of type B each day. So the profit P after selling these two products is given by the linear function:

$$P = 8x_1 + 5x_2 \quad \text{(Objective function)}$$

Since the company can produce at the most 500 hats in a day and A type of hats require twice as much time as that of type B, production restriction is given by

$$2tx_1 + tx_2 \leq 500t,$$

where t is the labour time per unit of second type i.e.

$$2x_1 + x_2 \leq 500$$

But, there are limitations on the sales of hats,

$$\therefore \; x_1 \leq 150 \;, \; x_2 \leq 250$$

LP model

$$\text{Maximize } P = 8x_1 + 5x_2$$

STC $\quad 2x_1 + x_2 \leq 500$

$$x_1 \leq 150$$
$$x_2 \leq 250$$
$$x_1, x_2 \geq 0$$

A manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredient available to make 20,000 bottles of medicines A and 40,000 bottles of medicine B, but there are only 45000 bottles into which either of the medicines can be filled. Further, it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for medicine A and Rs. 7 per bottle for medicine B. Formulate this problem as a LPP.

SOLUTION:

Suppose the manufacturer produces x_1 and x_2 thousand of bottles of medicines A and B, respectively. Since it takes three hours to prepare 1000 bottles of medicine A, the time required to fill x_1 thousand bottles of medicine A will be $3x_1$ hours. Similarly, the time required to prepare x_2 thousand bottles of medicine B will be x_2 hours. Therefore, total time required to prepare x_1 thousand bottles of medicine A and x_2 bottles of medicine B will be $3x_1 + x_2$ hours.

Now since the total time available for this operation is 66 hours.

$$3x_1 + x_2 \leq 66$$

Since there are only 45 thousand bottles available for filling medicines A and B, $x_1 + x_2 \leq 45$.

There are sufficient ingredients available to make 20 thousand bottles of medicine A and 40 thousand bottles of medicine B, hence $x_1 \leq 20$ and $x_2 \leq 40$

Number of bottles being non-negative, $x_1 \geq 0$, $x_2 \geq 0$

At the rate of Rs. 8 per bottle for type A medicine and Rs. 7 per bottle for type B medicine, the total profit on x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B will become:

$$P = 8 \times 1000\, x_1 + 7 \times 1000\, x_2$$

or $$P = 8000 x_1 + 7000 x_2$$

FORMULATION:

Max $P = 8000x_1 + 7000x_2$

STC $3x_1 + x_2 \leq 66$

$x_1 + x_2 \leq 45$

$x_1 \leq 20$

$x_2 \leq 40$

and $x_1, x_2 \geq 0$

- A toy company manufactures two types of dolls, a basic version - doll A and a delux version - doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day. The delux version requires a fancy dress of which there are only 600 per day available. If the Company makes a profit of Rs. 3.00 and Rs. 5.00 per doll, respectively on doll A and B, then how many of each doll should be produced per day in order to maximize total profit.

SOLUTION

Let x_1 and x_2 be the number of dolls produced per day of type A and B, respectively. Let the doll A require t hrs so that the doll B require 2t hrs. So the total time to manufacture x_1 and x_2 dolls shouldnot exceed 2000t hrs. Therefore, $tx_1 + 2tx_2 \le 2000t$.

Other constraints are simple.

LP model

$$
\begin{aligned}
\text{Maximize} \quad & P = 3x_1 + 5x_2 \\
\text{STC} \quad & x_1 + 2x_2 \le 2000 && \text{(time constraint)} \\
& x_1 + x_2 \le 1500 && \text{(plastic constraint)} \\
& x_2 \le 600 && \text{(dress constraint)} \\
\text{and} \quad & x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

- A firm can produce three types of cloth, say : A, B and C. Three kinds of wool are required for it, say : red, green and blue wool. One unit of length of type A cloth needs 2 metres of red wool and 3 metres of blue wool; one unit length of type B cloth needs 3 meters of red wool, 2 metres of green wool and 2 metres of blue wool; and one unit of type C cloth needs 5 metres of green wool and 4 metres of blue wool. The firm has only a stock of 8 metres of red wool, 10 metres of green wool and 15 metres of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3.00 of type B cloth is Rs. 5.00 and of type C cloth is Rs. 4.00

Determine, how the firm should use the available material so as to maximize the income from the finished cloth.

SOLUTION

Quality of Wool	Type of Cloth			Total quantity of wool available
	A(x_1)	B(x_2)	C(x_3)	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	Rs. 3.00	Rs. 5.00	Rs. 4.00	

Let x_1, x_2 and x_3 be the quantity produced of cloth type A, B, C respectively. Since 2 metres of red wool are required for each meter of cloth A and x_1 metres of this type of cloth are produced, so $2x_1$ meters of red wool will be required for cloth A. Similarly, cloth B requires $3x_2$ metres of red wool and cloth C doesnot require red wool. Thus, total quantity of red wool is

$$2x_1 + 3x_2 + 0x_3 \quad \text{(red wool)}$$

Similarly

$$0x_1 + 2x_2 + 5x_3 \quad \text{(green wool)}$$

$$3x_1 + 2x_2 + 4x_3 \quad \text{(blue wool)}$$

Since not more than 8 metres of red, 10metres of green and 15 metres of blue wool are available, the variables x_1, x_2 and x_3 must satisfy the following restrictions:

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

Also $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The total income of the finished cloth is given by:

(Max) $P = 3x_1 + 5x_2 + 4x_3$

DIET PROBLEM

One of the interesting problems in linear programming is that of balanced diet. Dieticians tell us that a balanced diet must contain quantities of nutrients such as calories, minerals, vitamins etc. Suppose that we are asked to find out the food that should be recommended from a large number of alternative sources of these nutrients so that the total cost of food satisfying minimum requirements of balanced diet is the lowest.

The medical experts and dieticians tell us that it is necessary for an adult to consume atleast 75g of proteins, 85g of fats, and 300g of carbohydrates daily. The following table gives the food items (which are readily available in the market), analysis, and their respective costs.

FOOD TYPE	FOOD VALUE (gms) per 100g			COST PER KG (Rs.)
	PROTEINS	FATS	CARBOHYDRATES	
1	8.0	1.5	35.0	1.00
2	18.0	15.0	–	3.00
3	16.0	4.0	7.0	4.00
4	4.0	20.0	2.5	2.00
5	5.0	8.0	40.0	1.50
6	2.5	–	25.0	3.00
MINIMUM DAILY REQUIREMENTS	7.5	85	300	

SOLUTION

Let x_1, x_2, x_3, x_4, x_5 and x_6 units of food types respectively be used per day in a diet and the total diet must atleast supply the minimum requirements. The object is to minimize total cost C of diet.

The objective function thus becomes

$$z = x_1 + 3x_2 + 4x_3 + 2x_4 + 1.5x_5 + 3x_6$$

Since 8, 18, 16, 4, 5 and 2.5 gms of proteins are available from 100gm unit of each type of food respectively, total proteins available from x_1, x_2, x_3, x_4, x_5 and x_6 units of each food respectively will be $8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6$ gms daily.

But minimum daily requirement of proteins as described is 75 gms. Hence, the protein requirements constraint is

$$8x_1 + 18x_2 + 16x_3 + 4x_4 + 5x_5 + 2.5x_6 \geqslant 75 \text{ (protein)}$$

Similarly, fats and carbohydrates requirement constraints are obtained respectively as below:

$$1.5x_1 + 15x_2 + 4x_3 + 20x_4 + 8x_5 + 0x_6 \geqslant 85 \text{ (fats)}$$

$$35x_1 + 0x_2 + 7x_3 + 2.5x_4 + 40x_5 + 25x_6 \geqslant 300 \text{ (carbohydrates)}$$

Also $x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0, x_5 \geqslant 0, x_6 \geqslant 0$.

- A manufacturer of biscuits is considering four types of gifts packs containing three types of biscuits : orange cream (OC), chocolate cream (CC) and wafers (W). Market research conducted recently to access the preferences of the consumers shows the following types of assortments to be in good demand:

ASSORTMENTS	CONTENTS	SELLING PRICE PER KG (Rs.)
A	Not less than 40% of OC, not more than 20% of CC, any qty of W	20
B	Not less than 20% of OC, not more than 40% of CC, any qty of W	25
C	Not less than 50% of OC, not more than 10% of CC, any qty of W	22
D	No restrictions	12

For the biscuits, the manufacturing capacity and costs are given:

Biscuits variety :	OC	CC	W
Plant capacity (kg/day) :	200	200	150
Manufacturing Cost (Rs./kg) :	8	9	7

Formulate a linear programming model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

SOLUTION :

Let the decision variables x_{ij} $(i = A, B, C, D ; j = 1, 2, 3)$ be defined as:

(i) For the gift pack A, x_{A1}, x_{A2}, x_{A3} denote the quantity in kg of OC, CC and W type of biscuits

(ii) For the gift pack B, x_{B1}, x_{B2}, x_{B3} denote the quantity in kg of OC, CC and W type of biscuits

(iii) For the gift pack C, x_{C1}, x_{C2}, x_{C3} denote the quantity in kg of OC, CC and W type of biscuits

(iv) For the gift pack D, x_{D1}, x_{D2}, x_{D3} denote the quantity ~~of~~ in kg of OC, CC and W type of biscuits

Now the given data can be put in the form of ~~the~~ LPP as:

$$\text{Maximize } P = 20(x_{A1} + x_{A2} + x_{A3}) + 25(x_{B1} + x_{B2} + x_{B3})$$

$$+ 22(x_{C1} + x_{C2} + x_{C3}) + 12(x_{D1} + x_{D2} + x_{D3})$$

$$- 8(x_{A1} + x_{B1} + x_{C1} + x_{D1})$$

$$- 9(x_{A2} + x_{B2} + x_{C2} + x_{D2})$$

$$- 7(x_{A3} + x_{B3} + x_{C3} + x_{D3})$$

$\therefore$ Maximize $P = 12x_{A1} + 11x_{A2} + 13x_{A3} + 17x_{B1} + 16x_{B2} + 18x_{B3}$
$+ 14x_{C1} + 13x_{C2} + 15x_{C3} + 4x_{D1} + 3x_{D2} + 5x_{D3}$

Subject to constraints

Gift pack A : $x_{A1} \geq 0.40(x_{A1} + x_{A2} + x_{A3})$
$x_{A2} \leq 0.20(x_{A1} + x_{A2} + x_{A3})$

Gift pack B : $x_{B1} \geq 0.20(x_{B1} + x_{B2} + x_{B3})$
$x_{B2} \leq 0.40(x_{B1} + x_{B2} + x_{B3})$

Gift pack C : $x_{C1} \geq 0.50(x_{C1} + x_{C2} + x_{C3})$
$x_{C2} \leq 0.10(x_{C1} + x_{C2} + x_{C3})$

Plant capacity constraints are:

$x_{A1} + x_{B1} + x_{C1} + x_{D1} \leq 200$

$x_{A2} + x_{B2} + x_{C2} + x_{D2} \leq 200$

$x_{A3} + x_{B3} + x_{C3} + x_{D3} \leq 150$

$x_{ij} \geq 0$ (for $i = A, B, C, D$ and $j = 1, 2, 3$)

- A complete unit of a certain product consists of four units of component A and three units of component B. Two component (A and B) are manufactured from two different raw materials of which 100 units and 200 units, respectively are available. Three departments are engaged in the production process with each department using a different method for manufacturing the components. The following table gives the raw material requirements per production run and the resulting units of each component. The objective is to determine the number of production runs for each department which will maximize the total number of components units of the final product.

Department	Input per run (units)		Output per run (units)	
	Raw material I & II		Component A	Component B
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

SOLUTION:

Let x_1, x_2, x_3 be the number of production runs for the departments 1, 2, 3 respectively.

The total number of units produced by three departments

$$6x_1 + 5x_2 + 7x_3 \quad \text{(component A)}$$

$$4x_1 + 8x_2 + 3x_3 \quad \text{(component B)}$$

The restrictions on the raw materials I and II are, respectively, given by

$$7x_1 + 4x_2 + 2x_3 \leq 100$$

$$5x_1 + 8x_2 + 7x_3 \leq 200$$

Since the objective function is to maximize the total number of units of the final product and each such unit requires 4 units of component A and 3 units of components B, the maximum number of units of the final product cannot exceed the smaller value of

$$\frac{1}{4}(6x_1 + 5x_2 + 7x_3) \quad \text{and} \quad \frac{1}{3}(4x_1 + 8x_2 + 3x_3)$$

The objective function thus becomes:

$$\text{Maximize} \quad Z = \min\left[\frac{1}{4}(6x_1+5x_2+7x_3), \frac{1}{3}(4x_1+8x_2+3x_3)\right]$$

Since this objective function is not linear, a suitable transformation can be used to reduce the above model to an acceptable linear programming format.

Suppose, $\min\left[\frac{1}{4}(6x_1+5x_2+7x_3), \frac{1}{3}(4x_1+8x_2+3x_3)\right] = v$

Therefore, $\frac{1}{4}(6x_1+5x_2+7x_3) \geq v$ and

$$\frac{1}{3}(4x_1+8x_2+3x_3) \geq v$$

In fact, at least one of these two inequalities must hold as an equation in any solution because the number of final assembly units, v, is maximized. Then, its upper limit is speified by the smaller of the left hand sides of above two inequalities. This indicates that the two inequalities are equivalent to the original equation defining v.

Now, the above problem can be put into the following linear programming form:

$$\text{Maximize} \quad Z = v$$

Subject to constraints

$$6x_1+5x_2+7x_3-4v \geq 0$$

$$4x_1+8x_2+3x_3-3v \geq 0$$

$$7x_1+4x_2+2x_3 \leq 100$$

$$5x_1+8x_2+7x_3 \leq 200$$

$$x_1, x_2, x_3, v \geq 0$$

- A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of cloths namely suitings, shirtings and woolens yielding the profit of Rs.2, Rs.4 and Rs.3 per metre respectively. One meter suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing 1 meter of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing while one meter woolen requires 3 minutes in each dept. In a week, total run time of each department is 60, 40 and 80 hours of weaving, processing and packing departments respectively. Formulate the LPP.

SOLUTION

Resources / Constraints	PRODUCT			Total Availability (minutes)
	SUITING	SHIRTING	WOOLEN	
Weaving Dept	3	4	3	60 X 60
Processing Dept	2	1	3	40 X 60
Packing Dept	1	3	3	80 X 60

LP model

Max (Total Profit) $P = 2x_1 + 4x_2 + 3x_3$

Subject to Constraints

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

A firm buys castings of P and Q type of parts and sells them as finished product after machining, boring and polishing. The purchasing cost for castings are Rs. 3 and Rs. 4 each for parts P and Q and selling costs are Rs. 8 and Rs. 10 respectively. The per hour capacity of machines used for machining, boring and polishing for two products is given below:

Capacity Per Hour	Parts	
	P	Q
Machining	30	50
Boring	30	45
Polishing	45	30

The running costs for machining, boring and polishing are Rs. 30, Rs. 22.5 and Rs. 22.5 per hour respectively. FORMULATE the LPP to find out the product mix to maximize the profit.

SOLUTION

Casting Operation	x_1	x_2
Machining	30/30 = 1.00	30/50 = 0.60
Boring	22.5/30 = 0.75	22.5/45 = 0.50
Polishing	22.5/45 = 0.50	22.5/30 = 0.75
Purchase	3	4
Total cost	5.25	5.85
Sale price	8	10
Profit	2.75	4.15

Let x_1, x_2 be the number of P and Q type parts to be produced per hour respectively. For the profit part of x_1 and x_2, we calculate the total cost for each and then subtract the sale price of that part from it.

On the machine, type of P parts consumes $1/30$ th of the available hour, a type Q part consumes $1/50$ th of an hour. Thus, the machine constraint becomes:

$$\frac{1}{30}x_1 + \frac{1}{50}x_2 \le 1 \quad \text{or} \quad 50x_1 + 30x_2 \le 1500$$

Similarly, other constraints can be established.

LP model

$$\text{Max (Total Profit) } P = 2.75x_1 + 4.15x_2.$$

Subject to the constraints

$$\frac{1}{30}x_1 + \frac{1}{50}x_2 \le 1$$

or $50x_1 + 30x_2 \le 1500$ (machine constraint)

$$\frac{1}{30}x_1 + \frac{1}{45}x_2 \le 1$$

or $45x_1 + 30x_2 \le 1350$ (boring constraint)

$$\frac{1}{45}x_1 + \frac{1}{30}x_2 \le 1$$

or $30x_1 + 45x_2 \le 1350$ (polishing constraint)

and $x_1, x_2 \geq 0$

- The owner of Fancy Goods shop is interested to determine how many advertisements to release in the selected three magazines, A, B and C. His main purpose is to advertise in such a way that total exposure to principal buyers of his goods is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements released multiplied by the number of principal buyers. The following data are available

PARTICULARS	MAGAZINES		
	A	B	C
Readers	1.0 lakh	0.6 lakh	0.4 lakh
Principal Buyers	20%	15%	8%
Cost per Adv.	8,000	6,000	5,000

The budgeted amount is at the most Rs.1.0 lakh for the advertisements. The owner has already decided that magazine A should have no more than 15 advertisements and that B and C each gets atleast 8 advertisements. Formulate LPP.

SOLUTION:

Let x_1, x_2 and x_3 be the required number of insertions in magazine A, B and C respectively. The total exposure of principal buyers of the magazine is:

$$Z = (20\% \text{ of } 1,00,000)x_1 + (15\% \text{ of } 60,000)x_2 + (8\% \text{ of } 40000)x_3$$

$$8000x_1 + 6000x_2 + 5000x_3 \leq 1,00,000 \quad \text{(budget constraints)}$$

$$x_1 \leq 15, \quad x_2 \geq 8 \text{ and } x_3 \geq 8 \quad \text{(advertisement constraint)}$$

The LP model is:

$$\text{Max (Total exposure) } P = 20000x_1 + 9000x_2 + 3200x_3$$

Subject to constraints:

$$8000x_1 + 6000x_2 + 5000x_3 \leq 1,00,000$$

$$0 \leq x_1 \leq 15, \quad 0 \leq x_2 \leq 8, \quad 0 \geq x_3 \geq 8$$

- Evening shift resident doctors in a Govt. hospital work five consecutive days and have two consecutive days off. Their five days of work can start on any day of the week and the schedule rotates indefinitely. The hospital requires the following minimum number of doctors working:

Sun	Mon	Tue	Wed	Thur	Fri	Sat
35	55	60	50	60	50	45

No more than 40 doctors can start their five working days on the same day. Formulate the general linear programming model to minimize the number of doctors employed by the hospital.

SOLUTION

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ be the number of doctors who start their duty on j^{th} $(j = 1, 2, \ldots 7)$ days of the week.

LP model

Max $\quad P = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

STC

$$x_1 + x_4 + x_5 + x_6 + x_7 \geqslant 35$$

$$x_2 + x_5 + x_6 + x_7 + x_1 \geqslant 55$$

$$x_3 + x_6 + x_7 + x_1 + x_2 \geqslant 60$$

$$x_4 + x_7 + x_1 + x_2 + x_3 \geqslant 50$$

$$x_5 + x_1 + x_2 + x_3 + x_4 \geqslant 60$$

$$x_6 + x_2 + x_3 + x_4 + x_5 \geqslant 50$$

$$x_7 + x_3 + x_4 + x_5 + x_6 \geqslant 45$$

$$0 \leq x_j \leq 40 \quad (j = 1, 2, \ldots 7)$$

GRAPHICAL METHOD

Step 1 : Convert each inequality into equality i.e. replace $\leq$ and $\geq$ by $=$

Step 2 : Convert each equation into the standard form

$$\frac{x}{a} + \frac{y}{b} = 1$$

which represents a straight line in intercept form. This line passes through $(a, 0)$ and $(0, b)$.

Step 3 : Find (shade) the feasible region

A feasible region is a region in which all the constraints hold good.

Step 4 : Substitute the co-ordinates of the corner points of the feasible region in the given objective function to optimize the given Linear Programming Problem (LPP).

Note

- If all the constraints are of the type $\le$, then feasible region will be towards the origin.
- When all the constraints are of the type $\ge$, then the feasible region will be away from the origin.
- x = constant, is a line parallel to y-axis.
- y = constant, is a line parallel to x-axis.
- $y = x$ is a straight line passing that the origin making an angle of 45° with the $+^{ve}$ x-axis.

- Maximize $z = x + 1.5y$

STC $x + 2y \leq 160$

$3x + 2y \leq 240$

$x, y \geqslant 0$

Solution

Line 1

$x + 2y = 160$

$\frac{x}{160} + \frac{y}{80} = 1$

$(160, 0)$ & $(0, 80)$

Line 2

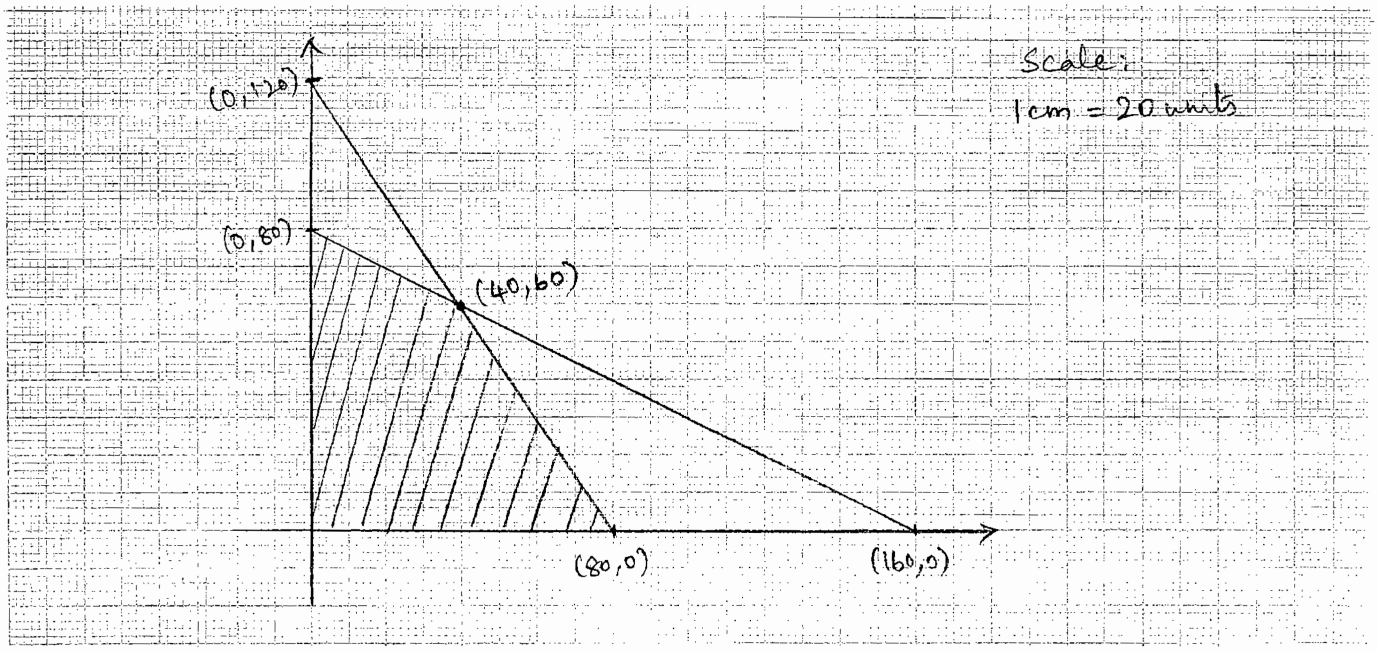
$3x + 2y = 240$

$\frac{x}{80} + \frac{y}{120} = 1$

$(80, 0)$ & $(0, 120)$

vertices	$z = x + 1.5y$
$(0, 0)$	0
$(80, 0)$	80
$(40, 60)$	130
$(0, 80)$	120

$Z_{max} = \underline{\underline{130}}$ and it occurs at $x = 40$ and $y = 60$



- Maximize $z = 3x + 5y$

  STC $x \leq 4$

  $2y \leq 12$

  $3x + 2y \leq 18$

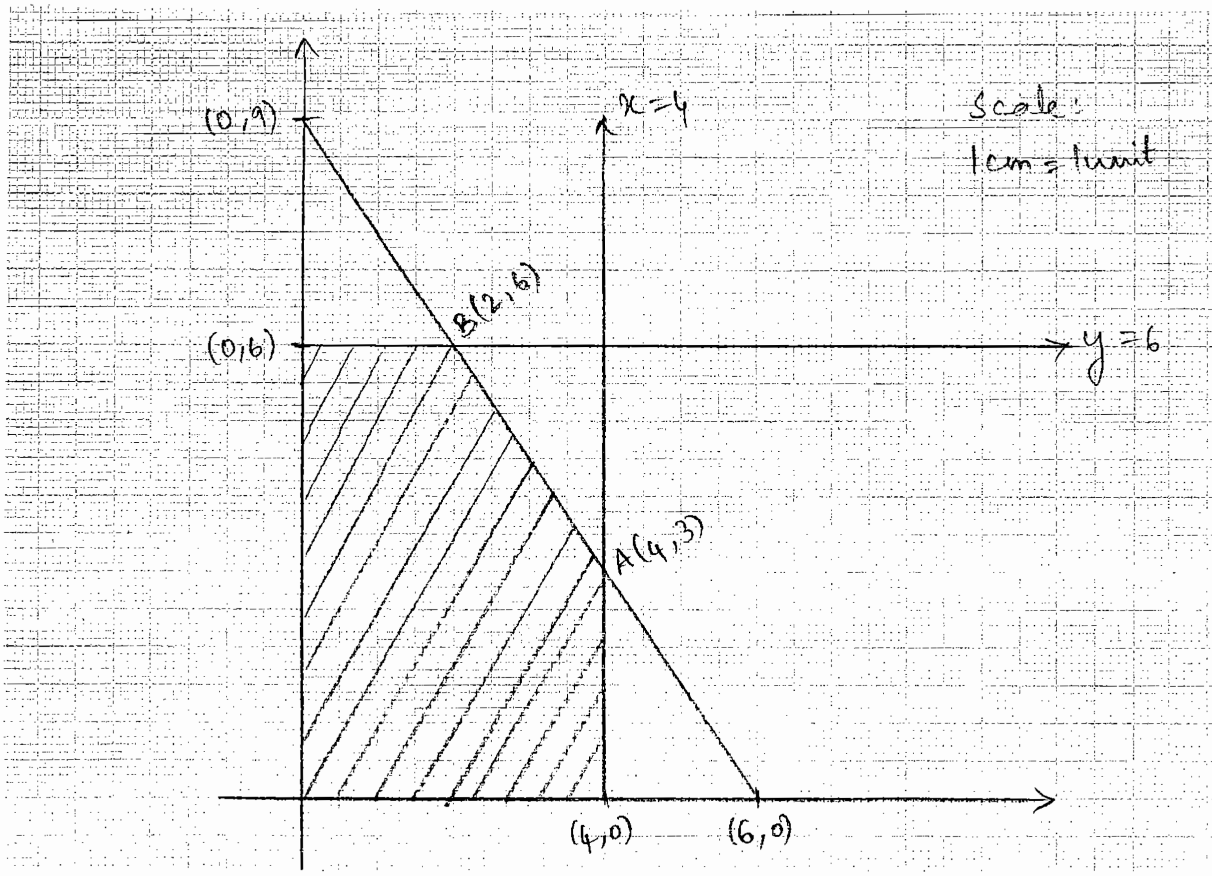
  where $x, y \geq 0$

Solution

Line 1	Line 2	Line 3
$x = 4$	$2y = 12$	$3x + 2y = 18$
$(4,0)$	$y = 6$	$\frac{x}{6} + \frac{y}{9} = 1$
	$(0,6)$	$(6,0)$ & $(0,9)$

vertices	$z = 3x + 5y$
$(0,0)$	0
$(4,0)$	12
$(4,3)$	27
$(2,6)$	36
$(0,6)$	30

$Z_{max} = \underline{\underline{36}}$ and it occurs at the point $x = 2$ & $y = 6$



- Minimize $z = 20x + 10y$

STC $x + 2y \leq 40$

$3x + y \geqslant 30$

$4x + 3y \geqslant 60$

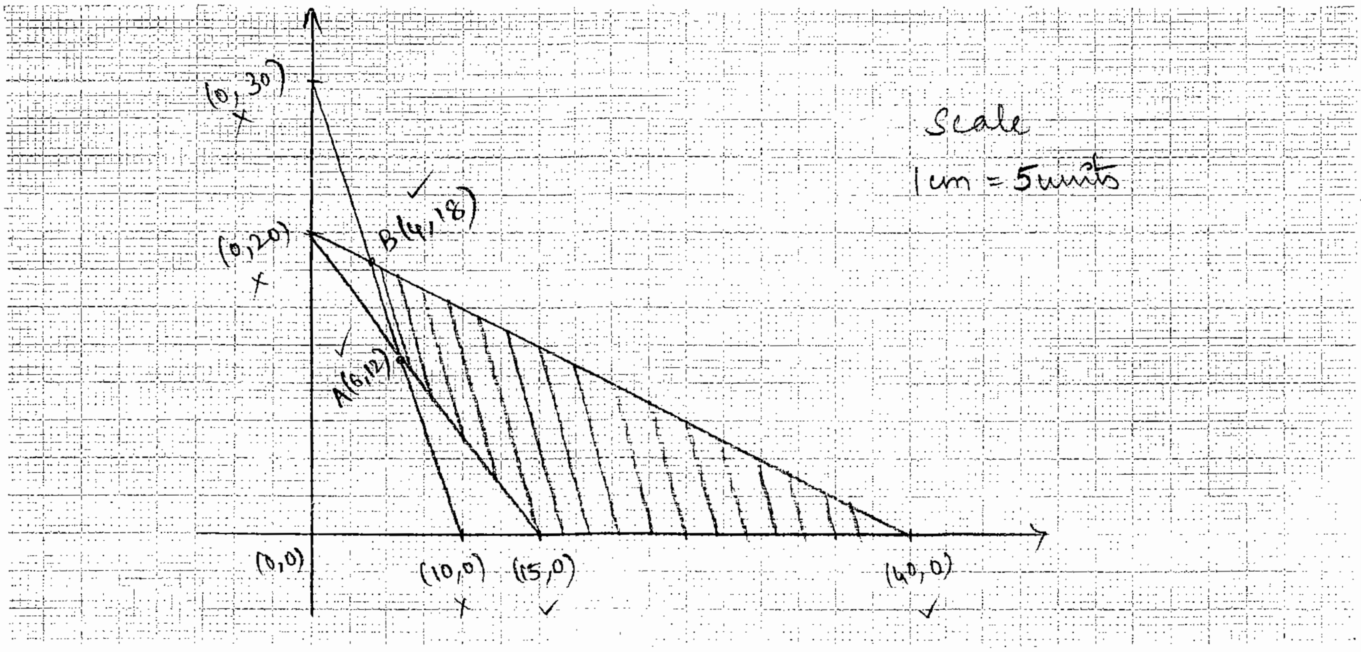
$x, y \geqslant 0$

Solution:

Line 1	Line 2	Line 3
$x + 2y = 40$	$3x + y = 30$	$4x + 3y = 60$
$\frac{x}{40} + \frac{y}{20} = 1$	$\frac{x}{10} + \frac{y}{30} = 1$	$\frac{x}{15} + \frac{y}{20} = 1$
(40, 0) & (0, 20)	(10, 0) & (0, 30)	(15, 0) & (0, 20)

vertices	$z = 20x + 10y$
(15, 0)	300
(40, 0)	800
(4, 18)	260
(6, 12)	240

$Z_{min} = \underline{\underline{240}}$ and it occurs at the point $x = 6$ and $y = 12$



- Maximize $z = 5x + 4y$

STC $6x + 4y \leq 24$

$x + 2y \leq 6$

$-x + y \leq 1$

$y \leq 2$

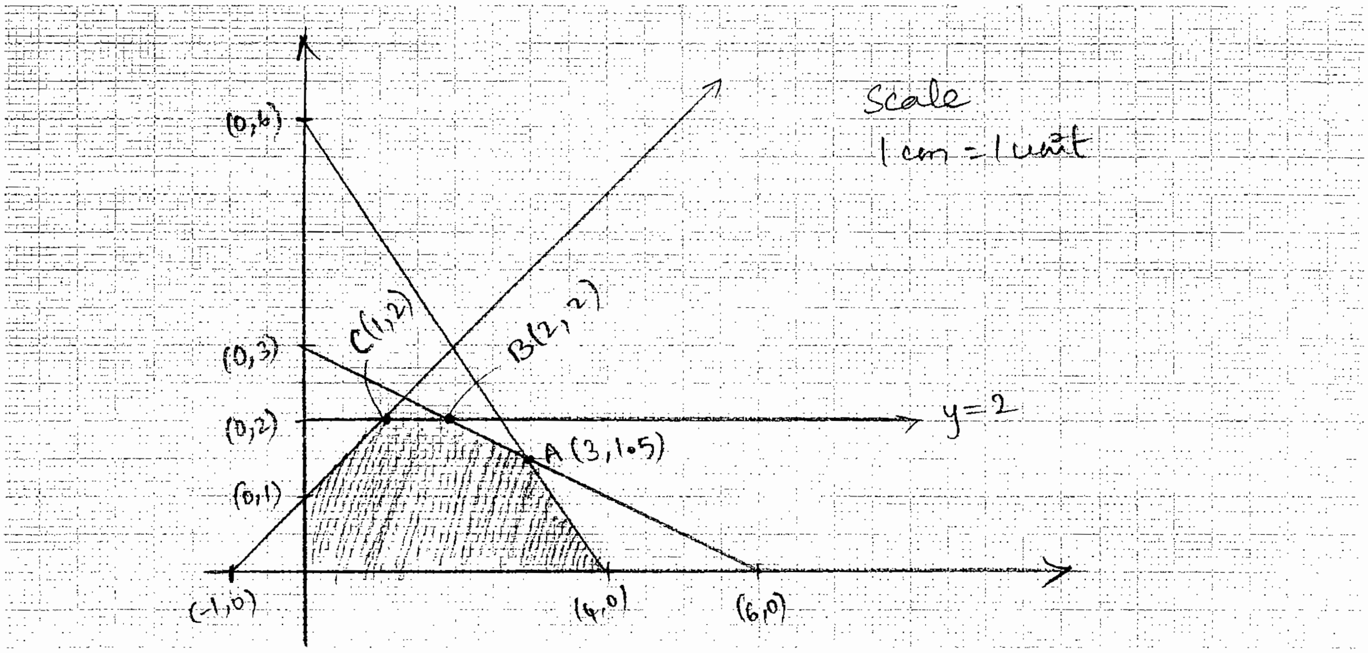
$x, y \geqslant 0$

Solution :

Line 1	Line 2	Line 3	Line 4
$6x + 4y = 24$	$x + 2y = 6$	$-x + y = 1$	$y \leq 2$
$\frac{x}{4} + \frac{y}{6} = 1$	$\frac{x}{6} + \frac{y}{3} = 1$	$\frac{x}{-1} + \frac{y}{1} = 1$	$(0,2)$
$(4,0)$ & $(0,6)$	$(6,0)$ & $(0,3)$	$(-1,0)$ & $(0,1)$	

vertices	$z = 5x + 4y$
$(0,0)$	0
$(4,0)$	20
A $(3,1.5)$	21
B $(2,2)$	18
C $(1,2)$	13
$(0,1)$	4

$Z_{max} = \underline{\underline{21}}$ and it occurs at the point $x = 3, y = 1.5$



- Maximize $z = 10x + 20y$

STC $-x + 2y \leq 15$

$x + y \leq 12$

$5x + 3y \leq 45$

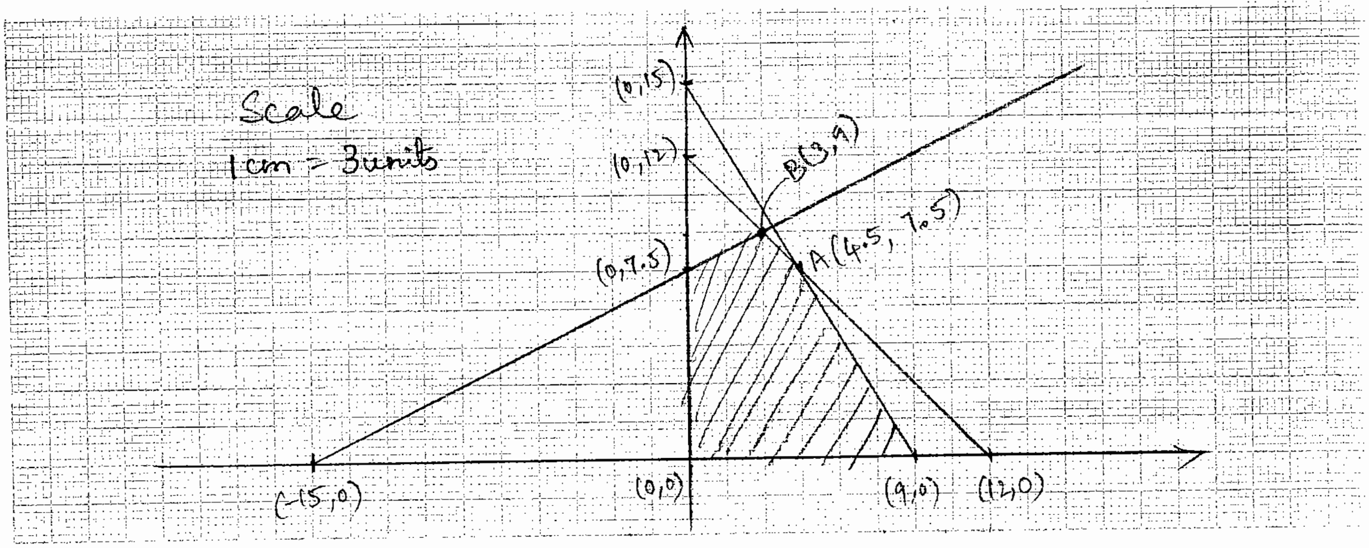
$x, y \geq 0$

Solution:

Line 1	Line 2	Line 3
$-x + 2y = 15$	$x + y = 12$	$5x + 3y = 45$
$\frac{x}{-15} + \frac{y}{7.5} = 1$	$\frac{x}{12} + \frac{y}{12} = 1$	$\frac{x}{9} + \frac{y}{15} = 1$
$(-15, 0)$ & $(0, 7.5)$	$(12, 0)$ & $(0, 12)$	$(9, 0)$ & $(0, 15)$

vertices	$z = 10x + 20y$
$(0,0)$	0
$(9,0)$	90
$(4.5, 7.5)$	195
$(3, 9)$	210
$(0, 7.5)$	150

$Z_{max} = \underline{\underline{210}}$ and it occurs at the point $x = 3$ and $y = 9$



- Maximize $Z = x + \frac{y}{2}$

STC $3x + 2y \leq 12$

$5x \leq 10$

$x + y \leq 18$

$-x + y \geq 4$

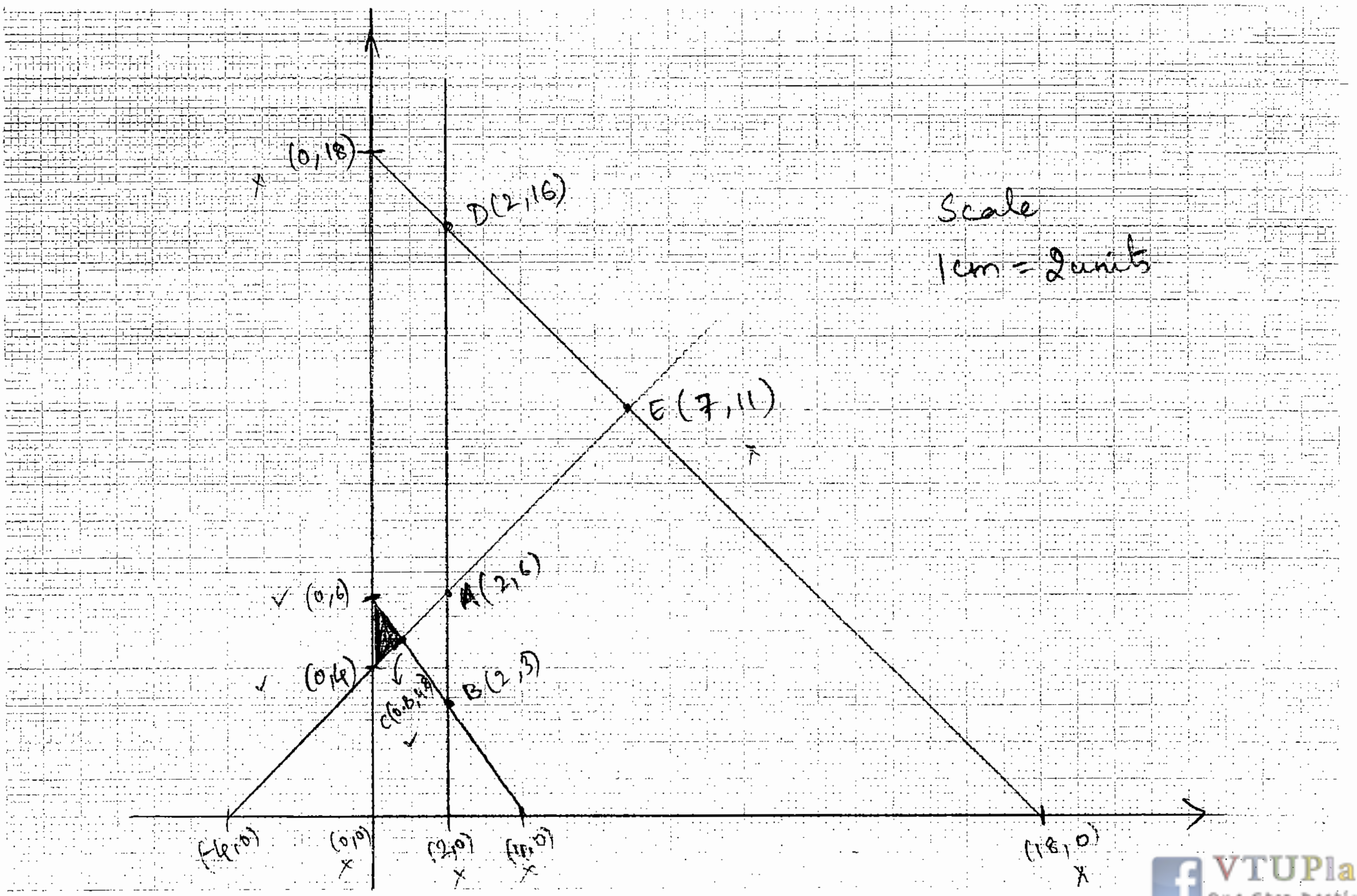
$x, y \geq 0$

Solution

<u>Line 1</u>	<u>Line 2</u>	<u>Line 3</u>	<u>Line 4</u>
$3x + 2y = 12$	$5x = 10$	$x + y = 18$	$-x + y = 4$
$\frac{x}{4} + \frac{y}{6} = 1$	$x = 2$	$\frac{x}{18} + \frac{y}{18} = 1$	$\frac{x}{-4} + \frac{y}{4} = 1$
(4,0) & (0,6)	(2,0)	(18,0) & (0,18)	(-4,0) & (0,4)

vertices	$Z = x + y/2$
(0,4)	2
(0, 6)	3
(0.8 , 4.8)	3.2

$Z_{max} = \underline{\underline{3.2}}$ and it occurs at $x = 0.8$ and $y = 4.8$



- Minimize $z = 5x + 5y$

STC $x + 2y \geqslant 10$

$x + y \geqslant 8$

$2x + y \geqslant 12$

$x, y \geqslant 0$

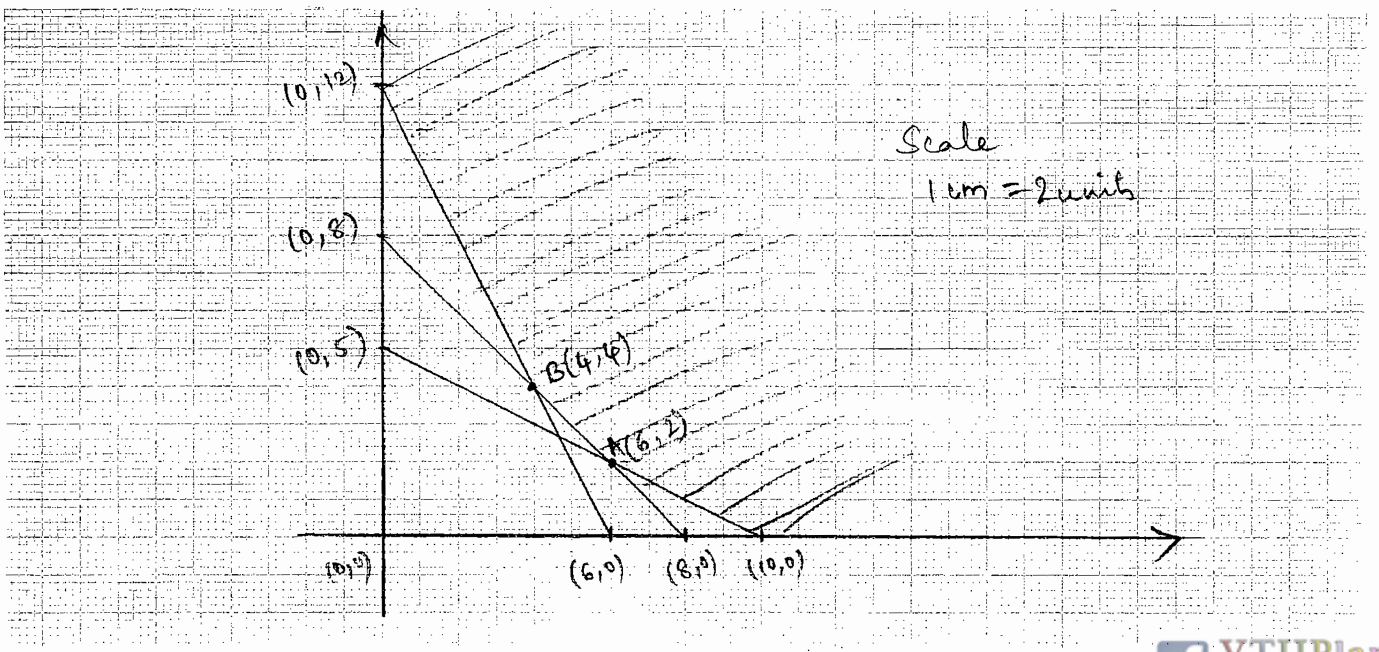
Solution

Line 1	Line 2	Line 3
$x + 2y = 10$	$x + y = 8$	$2x + y = 12$
$\frac{x}{10} + \frac{y}{5} = 1$	$\frac{x}{8} + \frac{y}{8} = 1$	$\frac{x}{6} + \frac{y}{12} = 1$
(10,0) & (0,5)	(8,0) & (0,8)	(6,0) & (0,12)

vertices	$Z = 5x + 5y$
(10,0)	50
(6,2)	40
(4,4)	40
(0,12)	60

We note that $Z_{min} = \underline{40}$ at (6,2) and (4,4)

∴ The value of Z is 40 at any point on the line joining A and B. As there are infinitely many number of points on the line AB, the given LPP has infinitely many number of solutions with minimum value = 40.



- Maximize $z = x + y$

STC $x + y \leq 1$

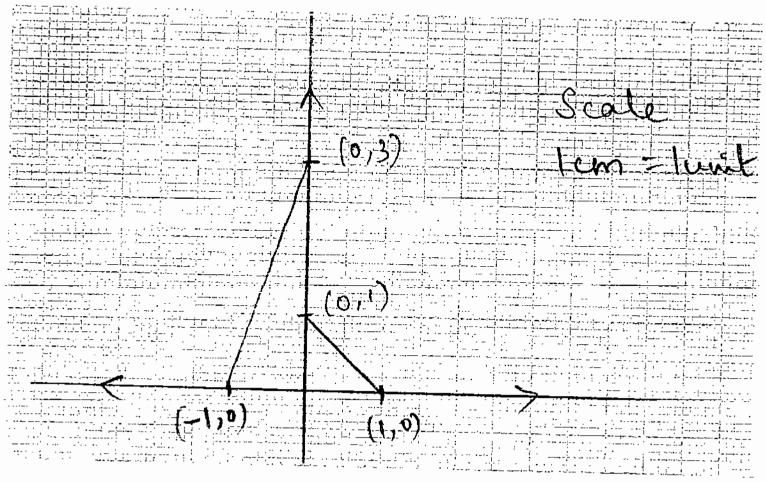
$-3x + y \geq 3$

$x, y \geq 0$

Solution:

Line 1	Line 2
$x + y = 1$	$-3x + y = 3$
$\frac{x}{1} + \frac{y}{1} = 1$	$\frac{x}{-1} + \frac{y}{3} = 1$
(1,0) & (0,1)	(-1,0) & (0,3)

We note that none of the points satisfy all the constraints. Therefore, feasible region doesnot exist and hence the given LPP cannot be solved by graphical method.



- Maximize $z = 100x + 40y$

STC $5x + 2y \leq 1000$

$3x + 2y \leq 900$

$x + 2y \leq 500$

$x, y \geq 0$

Solution

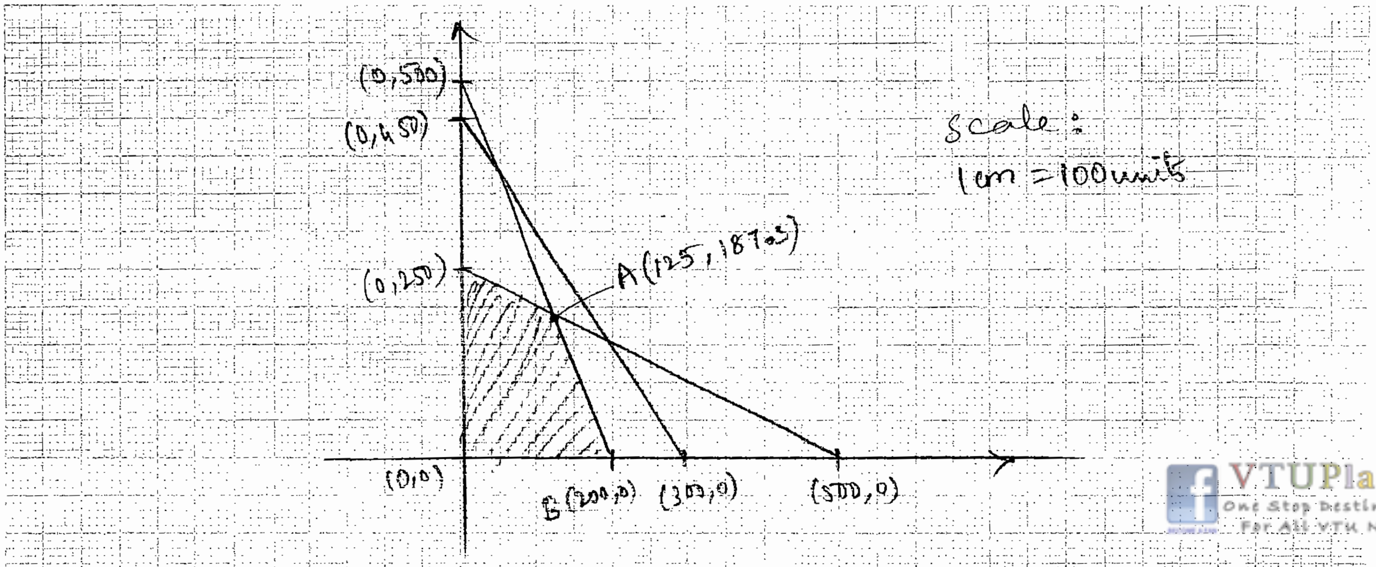
Line 1	Line 2	Line 3
$5x + 2y = 1000$	$3x + 2y = 900$	$x + 2y = 500$
$\frac{x}{200} + \frac{y}{500} = 1$	$\frac{x}{300} + \frac{y}{450} = 1$	$\frac{x}{500} + \frac{y}{250} = 1$
(200, 0) & (0, 500)	(300, 0) & (0, 450)	(500, 0) & (0, 250)

vertices	$z = 100x + 40y$
(0, 0)	0
B (200, 0)	20000
A (125, 187.5)	20000
(0, 250)	10000

We note that $z_{max} = 20000$ at B (200, 0) and A (125, 187.5)

∴ The value of z is 20000 at any point on the line joining A and B. But there are infinitely many points on the line

⇒ LPP has infinitely many numbers of solutions with maximum value of 20000.



- Minimize $z = 20x + 16y$

  STC $6x + 2y \geqslant 12$

  $2x + 2y \geqslant 8$

  $4x + 12y \geqslant 24$

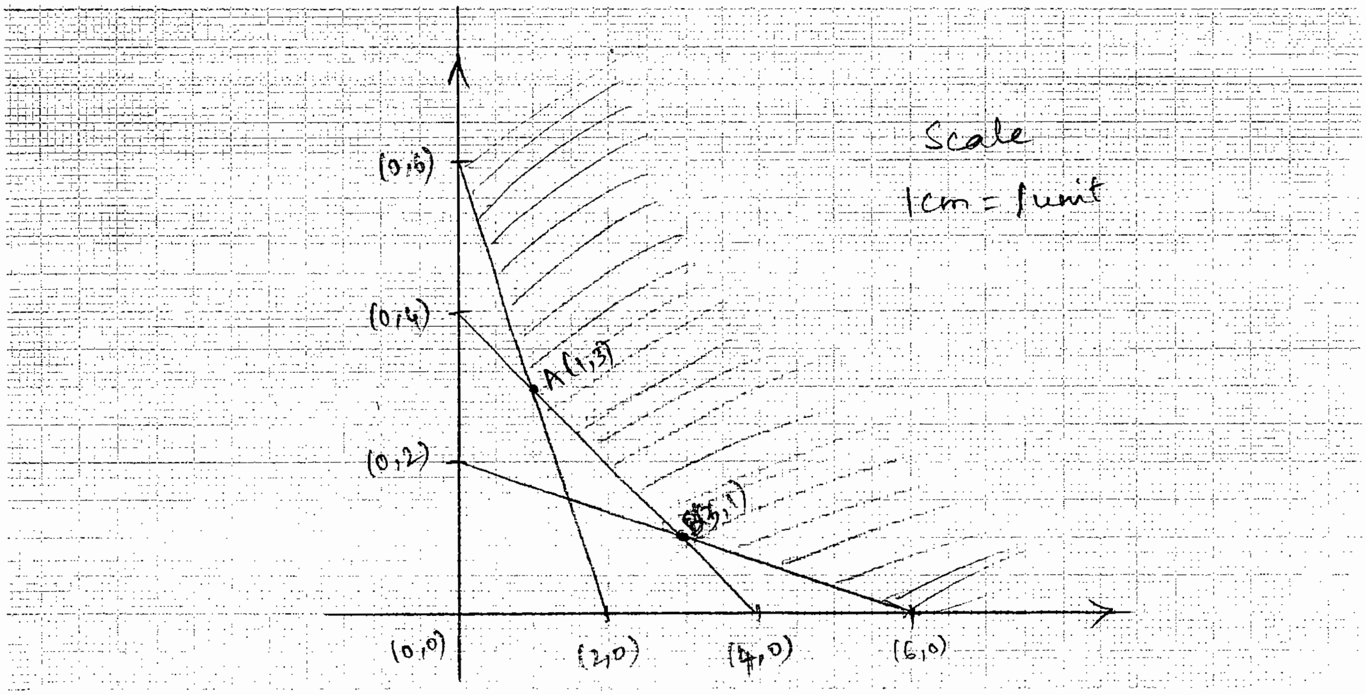
  $x, y \geqslant 0$

Solution:

Line 1	Line 2	Line 3
$6x + 2y = 12$	$2x + 2y = 8$	$4x + 12y = 24$
$\frac{x}{2} + \frac{y}{6} = 1$	$\frac{x}{4} + \frac{y}{4} = 1$	$\frac{x}{6} + \frac{y}{2} = 1$
(2, 0) & (0, 6)	(4, 0) & (0, 4)	(6, 0) & (0, 2)

vertices	$z = 20x + 16y$
(6, 0)	120
(3, 1)	76
(1, 3)	68
(0, 6)	96

$Z_{min} = \underline{\underline{68}}$ and occurs at the point $x = 1$ and $y = 3$



UNIT 2

SIMPLEX METHOD - I

The essence of the simplex Method

Simplex Method is an algebraic procedure. However, its underlying concepts are geometric. We can illustrate the general geometric concepts with the help of an example.

Example:

Maximize $z = 3x + 5y$

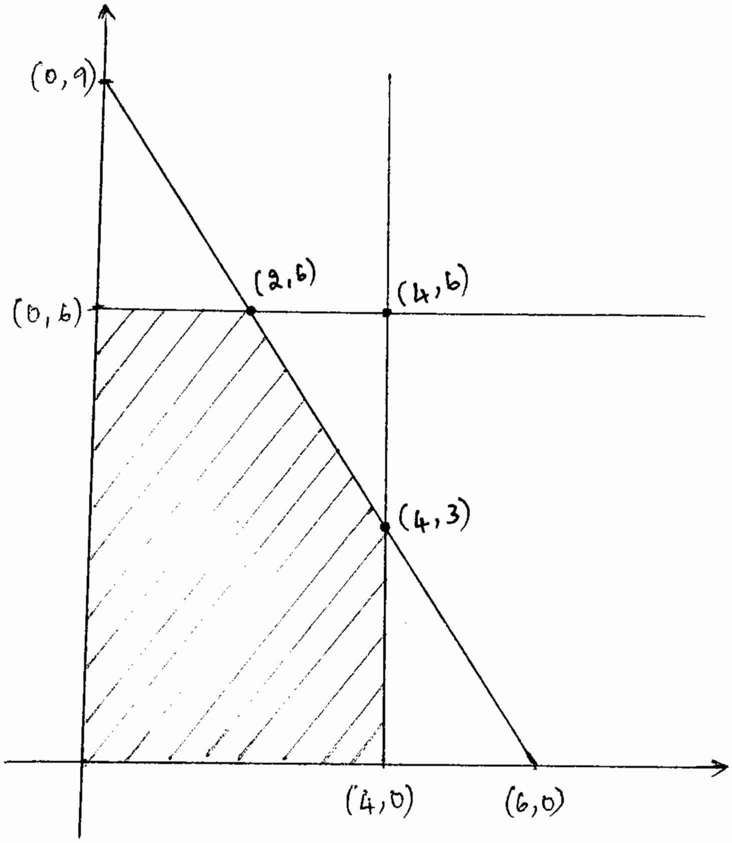
STC $x \leq 4$

$2y \leq 12$

$3x + 2y \leq 18$

where $x \geq 0, y \geq 0$

The graph for the above example is as shown below:



The five constraints and their points of intersection are highlighted in the graph. Here, each constraint boundary is a line that forms the boundary of what is permitted by the corresponding constraint. The points of intersection are the corner-point solutions of the problem. The points (0,0) (0,6), (2,6), (4,3) and (4,0) are the corner point feasible solutions (CPF solutions). The other points (0,9), (4,6) and (6,0) are called corner-point infeasible solution.

Optimality Test

Consider any LPP that processes at least one optimal solution. If a CPF solution has no adjacent CPF solⁿ that are better (as measured by objective function), then it must be an optimal solution.

Outline of Simplex Method from a geometric view point

Initialization:

Choose (0,0) as the initial CPF solution to examine (This is a convenient choice because no calculations are required to identify this CPF solution)

Optimality Test:

Conclude that (0,0) is not an optimal solution (adjacent CPF solutions are better)

Iteration I:

- Considering two edges of the feasible region that come-out from (0,0), choose to move along the edge that leads upto y-axis (with objective function $z = 3x + 5y$, moving up the y-axis increases z at a faster rate than moving along x-axis)

- Stop at the new constraint boundary $2y = 12$. [Moving farther in the direction selected in step 1 leaves the feasible region]
- Solve the intersection of the new set of boundaries to get (0, 6).

Optimality Test

Conclude that (0, 6) is not optimal.

(An adjacent CPF is better)

<u>Iteration II</u>

- Considering the two edges that come-out from (0, 6), choose to move along the edge that leads to the right
- Stop at the first new constraint boundary encountered when moving in that direction i.e. stop at $3x + 2y = 12$
- Solve the intersection of new set of constraint boundaries to get the point (2, 6)

Optimality Test

Conclude that (2, 6) is an optimal solution.

So, stop the procedure.

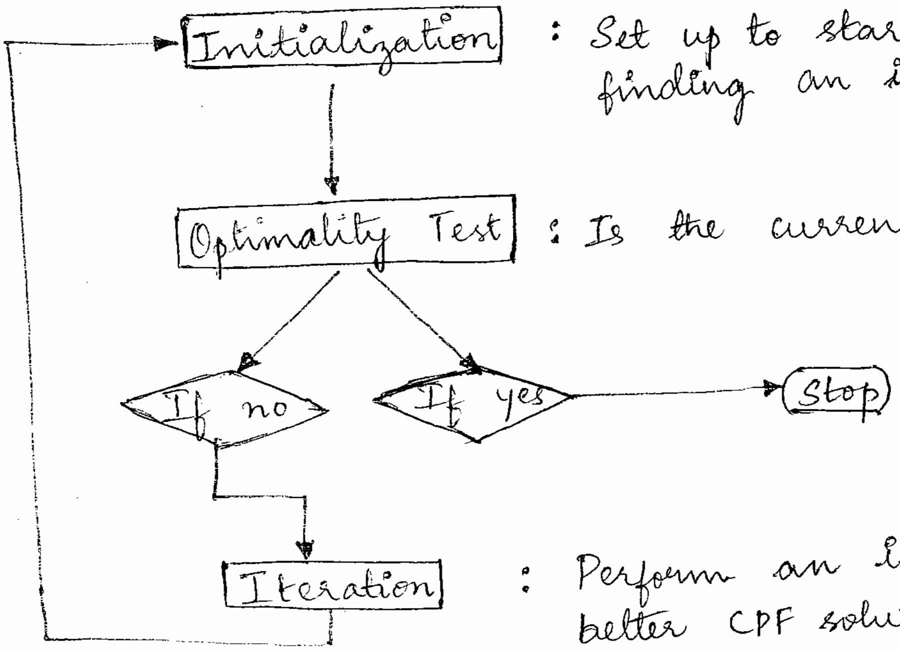
The key Solution Concepts

Solution Concept 1:

The Simplex method solely depends on CPF solutions. For any problem with atleast one optimal solution, finding one requires only finding a best CPF solution.

Solution Concept 2:

The Simplex method is an iterative algorithm with the following structure.



Initialization : Set up to start iterations, including finding an initial CPF solutions

Optimality Test : Is the current solution optimal?

Iteration : Perform an Iteration to find a better CPF solution.

Solution Concept 3:

Whenever possible, the initialization of the simplex method chooses the origin call decision variables equal to be the initial CPF solution. When there are too many decision variables to find an initial CPF solution graphically, the choice eliminates the need to use algebraic procedures to find and solve for an initial CPF solution.

Solution Concept 4

Given a CPF solution, it is much quicker computationally to gather information about its adjacent CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always choose a CPF that is adjacent to the current one. No other CPF solutions are considered. Consequently the entire path followed to eventually reach an optimal solution is along the edges of the feasible region.

Solution Concept 5

After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate (come-out) from this CPF solution. Each of these edges leads to an adjacent CPF solution at the other end, but the simplex method doesnot even take the time to solve for the adjacent CPF solution. Instead, it identifies the rate of improvement in Z (objective function) that would be obtained by moving along the edge. Among the edges with a positive rate of improvement in Z, it then chooses to move along the one with the largest rate of improvement in Z. The iteration is completed by first solving for the adjacent CPF solutions at the other end of this one edges and then relabeling this adjacent CPF solution as the current CPF solution for the optimality test and (if needed) the next iteration.

Solution Concept 6

Solution Concept 5 explains how simplex method examines each of the edges of the feasible region that come out from the CPF solution. This examination of an edge leads to quickly identifying the rate of improvement of Z. A positive rate of improvement in Z implies improvement in Z and a -ve rate of improvement in Z implies that the adjacent CPF solution is worse. Therefore the optimality test consists of checking whether any of the edges give a positive rate of improvement in Z. If none do, then the current CPF solution is optimal.

Assumptions of Linear Programming

Proportionality

The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j as represented by the $c_j x_j$ term in the objective function. Similarly, the contribution of each activity to the left-hand side of each functional constraint is proportional to the level of activity x_j as represented by a_{ij} x_i in the constraint.

Additivity Assumption

Every function in a LP~~P~~ model (whether the OF or the function on the LHS of a functional constraint) is the sum of the individual contributions of the respective activities.

Divisibility Assumption

Decision variables in a LP model are allowed to have any values including non integer values that satisfy the functional and non-ve constraints. Thus these variables are not restricted to just integer values. Since each decision variable represent the level of some activity it is being assumed that the activities can be run at fractional levels.

Certainity Assumption.

The values assigned to each parameter of a LP model is assumed to be a known constant.

Special Cases in Simplex Method (complicated situations)

We come across the following important difficulties while applying simplex method:

- Tie for key column
- Tie for key row (degeneracy)
- Unbounded problems
- Multiple optimal solutions
- Infeasible solution
- Redundant constraints
- Unrestricted variables

1. Tie for key column

When two negative (least) indicators become equal then there will be a tie for pivotal column. In such a situation, we select a column arbitrarily. There is no wrong choice, although selection of one variable may result in more iterations. Regardless of which variable column is selected, optimal solution will be same.

2. Tie for pivotal row

When two non -ve ratios (least) are equal, there will be a tie for pivotal row. Such a situation in simplex method is known as degeneracy. In such a case, either pivotal row is selected arbitrarily or selected using the following rule known as perturbation rule with following steps:

→ Locate rows in which smallest non -ve ratios are tied.

→ Find the co-efficient of the slack variables and divide each coefficient by the corresponding positive numbers of the key column in the row starting from left to right.

→ Compare the resulting ratios column by column.

→ Select the row with smallest ratio as pivotal row.

→ After resolving tie, simplex method is applied as usual.

3. Unbounded problems

If all the ratios are -ve, then we can't select pivotal row. In such a case, we say that the problem has an unbounded solution.

4. Multiple Optimal Solution

In the final simplex table, if the indicator value of all decision variables are zeros and if a slack or surplus variable enters the optimum basis, then we say that LPP has multiple optimal solution.

5. Infeasible Solutions

In the final simplex table, if artificial variable is present in the optimum basis at a positive level, then LPP has an infeasible solution.

6. Redundant Constraints

Suppose a LPP has 2 constraints like $3x + 4y \leq 7$ and $3x + 4y \leq 15$. These two are called redundant constraints. These redundant constraints increase the computational work unnecessarily.

7. Unrestricted Variables:

If a LPP has un-restricted variables then each unrestricted variable is expressed as difference of two non -ve variables. This will increases the computational work considerably.

UNIT 2

SIMPLEX METHOD

It is an iterative method to solve a linear programming problem.

Working Rule:

- Maximize $z = ax + by$

STC $a_1 x + b_1 y \leq k_1$

$a_2 x + b_2 y \leq k_2$

Step 1:

Introduce 2 slack variables s_1 and s_2

(Slack variable is a variable which converts $\leq$ to $=$)

(Surplus variable is a variable which converts $\geq$ to $=$)

$a_1 x + b_1 y + s_1 = k_1$ ——①

$a_2 x + b_2 y + s_2 = k_2$ ——②

Step 2: The above equations are put in the form of a table and the table thus obtained is called as initial simplex table

	coefficients in OF →		a	b	0	0
NZV	C_B	X_B	x	y	S_1	S_2
S_1	0	k_1	a_1	b_1	1	0
S_2	0	k_2	a_2	b_2	0	1
$z_j - c_j$	$Z = C_B X_B$		Δ_1	Δ_2	Δ_3	Δ_4

C_B = Coefficient of basic variables

(NZV → non-zero variables or Basic variables)

($z_j - c_j$ = indicators)

The first two rows of the above table contains various co-efficients from equations ① & ②

$Z = C_B X_B = (0 \times k_1) + (0 \times k_2) = 0$

$\Delta_1 = C_B X - a$

$= (0 \times a_1 + 0 \times a_2) - a$

$= -a$

$\Delta_2 = C_B y - b$

$= (0 \times b_1 + 0 \times b_2) - b$

$= -b$

$\Delta_3 = (C_B S_1) - 0$

$= (0 \times 1 + 0 \times 0) - 0$

$= 0$

$\Delta_4 = (C_B S_2) - 0$

$= (0 \times 0 + 0 \times 1) - 0$

$= 0$

In the initial simplex table, the last row will have entries which are negative of the ~~obj~~ coefficients in OF. They are called as indicators.

Step 3 : If all Δ values are non negative, then the current solution is the optimal solution. If atleast one Δ is negative, then the solution is not optimal and hence go to step 4

Step 4 : (i) Identify the least negative indicator. The column in which it is present is called as pivotal column.

(ii) Divide X_B column by pivotal column to get ratios. Identify the least non-negative ratio (ignore -ve ratios & ignore -0 also)

The row in which the least non-negative ratio is present is called as the pivotal row.

The entry which is common to both pivotal row and pivotal column is called as pivotal element or key element.

The variable corresponding to pivotal row is called as Outgoing variable (departing variable) and the variable corresponding to pivotal column is called as incoming variable (arriving variable).

(iii) Make the pivotal element unity by dividing pivotal row by pivotal element

(iv) Make the other entries of the pivotal column zero by appropriate elementary row operations. (column operations are not allowed)

Step 5 : For the new simplex table, repeat steps 3 & 4 until all indicators become non-negative.

PROBLEMS : Solve by Simplex Method

1. Maximize $z = x + 1.5y$

STC $x + 2y \leq 160$

$3x + 2y \leq 240$

$x, y \geq 0$

Solution:

Introduce S_1 and S_2

$x + 2y + S_1 = 160$

$3x + 2y + S_2 = 240$

			1	1.5	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	Ratios
S_1	0	160	1	②	1	0	$160/2 = 80$ ← PR
S_2	0	240	3	2	0	1	$240/2 = 120$
Δ	$Z = 0$		-1	-1.5 ↑ PC	0	0	–

Pivot = 2

OGV = S_1

ICV = y

Apply $R_1' = R_1 \div 2$

$R_2' = R_2 - 2R_1'$

			1	1.5	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	Ratios
y	1.5	80	$1/2$	1	$1/2$	0	$80/\tfrac{1}{2} = 160$
S_2	0	80	②	0	-1	1	$80/2 = 40$ ← PR
Δ	$Z = 120$		$-1/4$ ↑ PC	0	$3/4$	0	

Pivot = 2

OGV = S_2

ICV = x

Apply $R_2' \rightarrow R_2/2$

$R_1' \rightarrow R_1 - \tfrac{1}{2}R_2'$

			1	1.5	0	0
NZV	C_B	X_B	x	y	S_1	S_2
y	1.5	60	0	1	$3/4$	$-1/4$
x	1	40	1	0	$-1/2$	$1/2$
Δ	$Z = 130$		0	0	$5/8$	$1/8$

As all the indicators are non -ve, $Z_{max} = 130$ and it occurs at $x = 40$ and $y = 60$

2) Using simplex method, maximize $Z = 2x + 4y$

STC $3x + y \leq 22$

$2x + 3y \leq 24$

$x, y \geq 0$

Solution: Introduce slack variables s_1 and s_2

$$3x + y + S_1 = 22$$

$$2x + 3y + S_2 = 24$$

			2	4	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	ratios
S_1	0	22	3	1	1	0	$22/1 = 22$
S_2	0	24	2	③	0	1	$24/3 = 8$ ←
Ind	$Z = 0$		-2	-4 ↑	0	0	–

Pivotal element is 3

Outgoing variable = S_2 Apply $R_2' \to R_2/3$

incoming variable = y $R_1' \to R_1 - R_2'$

			2	4	0	0
NZV	C_B	X_B	x	y	S_1	S_2
S_1	0	14	$7/3$	0	1	$-1/3$
y	4	8	$2/3$	1	0	$1/3$
Ind		$Z = 32^*$	$2/3$	0	0	$4/3$

As all the indicators are non-negative, simplex procedure is complete. $\therefore Z_{max} = 32$ at $x = 0$, $y = 8$

3) Using simplex method, maximize $Z = 3x + 4y$

STC $2x + y \leq 40$

$2x + 5y \leq 180$

$x, y \geq 0$

Solution: $2x + y + S_1 = 40$

$2x + 5y + S_2 = 180$

			3	4	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	Ratios
S_1	0	40	2	1	1	0	$40/1 = 40$
S_2	0	180	2	$\boxed{5}$	0	1	$180/5 = 36$ ←
Ind		$Z=0$	-3	-4 ↑	0	0	←

Pivotal element = 5 Apply $R_2' \to R_2/5$

O.g.v = S_2 $R_1' \to R_1 - R_2'$

i.c.v = y

			3	4	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	Ratios
S_1	0	4	$\boxed{8/5}$	0	1	$-1/5$	$4/8/5 = 20/8 = 2.5$ ←
y	4	36	$2/5$	1	0	$1/5$	$36/2/5 = 180/2 = 90$
Ind		$Z=144$	$-7/5$ ↑	0	0	$4/5$	

Pivotal element = $8/5$ Apply $R_1' \to R_1 \times 5/8$

outgoing variable = S_1 $R_2' \to R_2 - 2/5\, R_1'$

Incoming variable = x

NZV	C_B	X_B	x	y	S_1	S_2
x	3	$5/2$	1	0	$5/8$	$-1/8$
y	4	35	0	1	$-1/4$	$1/4$
Ind		$Z=147.5$	0	0	$7/8$	$5/8$

As all the indicators are non-negative, simplex procedure is complete.

$\therefore Z_{max} = 147.5$ at $x = 5/2$, $y = 35$

4). Using simplex method, maximize $p = 5x + 8y$

STC $4x + 6y \leq 24$

$2x + y \leq 18$

$3x + 9y \leq 36$

<u>Solution</u> : Introduce 3 slack variables s_1, s_2 and s_3

			5	8	0	0	0			
NZV	C_B	X_B	x	y	S_1	S_2	S_3	Ratios	New Ratios	
S_1	0	24	4	6	1	0	0	$24/6 = 4$	$1/6 = 0.1667$	
S_2	0	18	2	1	0	1	0	$18/1 = 18$	–	
S_3	0	36	3	⑨	0	0	1	$36/9 = 4$	$0/9 = 0$	⟵
Ind	$Z = 0$		-5	-8 ↑	0	0	0	–	–	

We note that there is a tie for pivotal row b/w 1st and 3rd rows. Such a situation in simplex method is known as degeneracy in simplex method. We can break the tie arbitrarily or we can use perturbation rule to identify pivotal row.

Out of 0 and 0.1667, 0 is least.

∴ Pivotal element is 9.

O.G.V = S_3

I.C.V = y

Apply $R_3' \rightarrow R_3/9$

$R_1' \rightarrow R_1 + 6R_3'$

$R_2' \rightarrow R_2 - R_3'$

NZV	C_B	X_B	x	y	S_1	S_2	S_3	Ratios	
S_1	0	0	②	0	1	0	$-2/3$	0	⟵
S_2	0	14	$5/3$	0	0	1	$-1/9$	$42/5$	
y	8	4	$1/3$	1	0	0	$1/9$	12	
Ind	$P = 32$		↑	0	0	0	$8/9$		

pivotal element is 2 Apply $R_1' \to R_1/2$

O.G.V = S_1 $R_2' \to R_2 - 5/3 R_1'$

I.C.V = x $R_3' \to R_3 - 1/3 R_1'$

NZV	C_B	X_B	x	y	S_1	S_2	S_3
x							
S_2							
y							
Ind							

$P_{max} = 32$, at $x = 0$, $y = 4$

5) Maximize $P = 60x + 50y$

STC $4x + 2y \le 80$

$3x + 2y \le 60$

$x, y \ge 0$

$P_{max} = 50$ at $x = 0$, $y = 30$

Trial and Error Method

1) Maximize $z = x_1 + 3x_2 + 3x_3$

STC $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$

Also find which of the basic solutions are

a) basic feasible

b) non-degenerate basic feasible

c) optimal basic feasible.

Solution :

We have 3 unknowns and 2 equations

∴ Total no of basic solutions = $^3C_2$ = 3

Set (3 - 2) = 1 unknown as zero

~~Sl.No. Non basic variables Basic variables Equations~~

Sl No	Non basic variables	Basic variables	Equations	Solⁿ	Basic Feasibility	Non deg	Value of OF	Optimal solⁿ
1	$x_1 = 0$	x_2, x_3	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$	$x_1 = 0$ $x_2 = -1$ $x_3 = 2$	No	No	-	-
2	$x_2 = 0$	x_1, x_3	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$	$x_1 = 1$ $x_2 = 0$ $x_3 = 1$	Yes	No	4	No
3	$x_3 = 0$	x_1, x_2	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$	$x_1 = 2$ $x_2 = 1$ $x_3 = 0$	Yes	No	5	Yes

a) All the three are basic feasible

b)

c) Solution 3 is optimal

2) Maximize $Z = 12x_1 + 8x_2 + 14x_3 + 10x_4$

STC $5x_1 + 4x_2 + 2x_3 + x_4 = 100$

$2x_1 + 3x_2 + 8x_3 + x_4 = 75$

Solution:

Given system has 4 unknowns and two equations

$\therefore$ total no of basic solutions possible $= 4C_2 = 6$

Set $(4-2) = 2$ unknowns as zero at a time

Sl.No.	Non basic variables	Basic variables	Equations	Solution	Basic Feasibility	Value of OF ~~Optimal Soln~~	Optimal solⁿ
1	$x_1 = 0$ $x_2 = 0$	x_3 x_4	$2x_3 + x_4 = 100$ $8x_3 + x_4 = 75$	$x_1 = 0$ $x_2 = 0$ $x_3 = -4.167$ $y = 108.33$	No	-	-
2	$x_1 = 0$ $x_3 = 0$	x_2 x_4	$4x_2 + x_4 = 100$ $3x_2 + x_4 = 75$	$x_1 = 0$ $x_2 = 25$ $x_3 = 0$ $x_4 = 0$	Yes	200	No
3	$x_1 = 0$ $x_4 = 0$	x_2 x_3	$4x_2 + 2x_3 = 100$ $3x_2 + 8x_3 = 75$	$x_1 = 0$ $x_2 = 25$ $x_3 = 0$ $x_4 = 0$	Yes	200	No
4	$x_2 = 0$ $x_3 = 0$	x_1 x_4	$5x_1 + x_4 = 100$ $2x_1 + x_4 = 75$	$x_1 = 8.33$ $x_2 = 0$ $x_3 = 0$ $x_4 = 58.33$	Yes	683.26	Yes
5	$x_2 = 0$ $x_4 = 0$	x_1 x_3	$5x_1 + 2x_3 = 100$ $2x_1 + 8x_3 = 75$	$x_1 = 18.056$ $x_2 = 0$ $x_3 = 4.861$ $x_4 = 0$	Yes	284.72	No

	NBV	BV	Eq	Soln	BF	Value of OF	Optimal soln
6	$x_3 = 0$	x_1	$5x_1 + 4x_2 = 100$	$x_1 = 0$			
	$x_4 = 0$	x_2	$2x_1 + 3x_3 = 75$	$x_2 = 25$	Yes	200	No
				$x_3 = 0$			
				$x_4 = 0$			

3) Find all the basic solutions of the following system of equations identifying in each case the basic & non-basic variables

July 2009

$$2x_1 + x_2 + 4x_3 = 11$$

$$3x_1 + x_2 + 5x_3 = 14$$

Solution :

Given system has 2 equations & 3 unknowns

$\therefore$ total no. of basic solutions $= {}^3C_2 = 3$

Set $(3-2) = 1$ unknown to zero at a time

Sl.No	Non basic variables	Basic variables	Equations	Soln
1	$x_1 = 0$	x_2	$x_2 + 4x_3 = 11$	$x_1 = 0$
		x_3	$x_2 + 5x_3 = 14$	$x_2 = -1$
				$x_3 = 3$
2	$x_2 = 0$	x_1	$2x_1 + 4x_3 = 11$	$x_1 = 1/2$
		x_3	$3x_1 + 5x_3 = 14$	$x_2 = 0$
				$x_3 = 5/2$
3	$x_3 = 0$	x_2	$2x_1 + x_2 = 11$	$x_1 = 3$
		x_1	$3x_1 + x_2 = 14$	$x_2 = 5$
				$x_3 = 0$

4) Find all basic solutions of the following system of equations identifying in each case the basic & non-basic variables

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Solution

Given system has 2 equations & 4 unknowns

∴ total no of basic solutions $= {}^4C_2 = 6$

Set $(4-2) = 2$ unknowns to zero at a time

Sl.No	Non basic variables	Basic variables	Equations	solution
1	$x_1 = 0$ $x_2 = 0$	x_3 x_4	$6x_2 + 2x_3 = 3$ $4x_2 + 4x_3 = 2$	
2	$x_1 = 0$ $x_3 = 0$	x_2 x_4	$6x_2 + x_4 = 3$ $4x_2 + 6x_4 = 2$	
3	$x_1 = 0$ $x_4 = 0$	x_2 x_3	$6x_2 + 2x_3 = 3$ $4x_2 + 4x_3 = 2$	
4	$x_2 = 0$ $x_3 = 0$	x_1 x_4	$2x_1 + x_4 = 3$ $6x_1 + 6x_4 = 2$	
5	$x_2 = 0$ $x_4 = 0$	x_1 x_3	$2x_1 + 2x_3 = 3$ $6x_1 + 4x_3 = 2$	

6	$x_3=0$	x_1	$2x_1+6x_2=3$
	$x_4=0$	x_2	$6x_1+4x_2=2$

HW

⑤ Maximize $z = 3x + 2y$ June 2010

STC $x + y \leq 4$

$x - y \leq 2$

$x, y \geq 0$

⑥ Maximise $z = 6x + 8y$

STC $2x + 8y \leq 16$

$2x + 4y \leq 8$

$x, y \geq 0$

Simplex Method

⑦ Max $P = 4x + 3y + 6z$

July 2009

STC $2x + 3y + 2z \leq 440$

$4x + 3z \leq 470$

$2x + 5y \leq 430$

Solution:

			4	3	6	0	0	0	
NZV	C_B	X_B	x	y	z	S_1	S_2	S_3	Ratios
S_1	0	440	2	3	2	1	0	0	220
S_2	0	470	4	0	③	0	1	0	470/3 ←
S_3	0	430	2	5	0	0	0	1	~~215~~ ∞
$Z_j = C_j$		P = 0	-4	-3	-6	0	0	0	
					↑				

Pivot = 3 Apply $R_2' \rightarrow R_2/3$

OGV = S_2 $R_1' \rightarrow R_1 - 2R_2'$

ICV = z

NZV	C_B	X_B	x	y	z	S_1	S_2	S_3	Ratios
S_1	0	$380/3$	$-2/3$	$\boxed{3}$	0	1	$-2/3$	0	$380/9$ ←
z	6	$470/3$	$4/3$	0	1	0	$1/3$	0	∞
S_3	0	430	2	5	0	0	0	1	$430/5$
$Z_j - C_j$	$P = 940$		4	-3 ↑	0	0	2	0	–

Pivot = 3

OGV = S_1

ICV = y

Apply $R_1' \to R_1/3$

$R_3' \to R_3 - 5R_1'$

NZV	C_B	X_B	x	y	z	S_1	S_2	S_3
y	3	$380/9$	$-2/9$	1	0	$1/3$	$-2/9$	0
z	6	$470/3$	$4/3$	0	1	0	$1/3$	0
S_3	0	$1970/9$	$28/9$	0	0	$-5/3$	$10/9$	1
$Z_j - C_j$	$P = \frac{3200}{3}$		$10/3$	0	0	1	$4/3$	0

As all indicators are non-negative, simplex procedure is complete.

$\therefore P_{max} = \frac{3200}{3}$ at $x = 0$

$y = \frac{380}{9}, \quad z = \frac{470}{3}$

Using simplex method, minimize $P = x - 3y + 3z$

Dec 2010

STC $3x - y + 2z \leq 7$

$\boxed{2x + 4y \geq -12}$

$-4x + 3y + 8z \leq 10$

Solution:

Simplex method will always maximize the objective function. We know minimization = − (maximization)

In this problem, we shall maximize $P' = -x + 3y - 3z$

We shall convert 2nd inequality into an inequality of type ≤

by multiplying throughout by -1. Therefore, the constraints are

$$3x - y + 2z \le 7$$
$$-2x - 4y \le 12$$
$$-4x + 3y + 8z \le 10$$

~~NZV C_B X_B x y z S_1 S_2 S_3~~

Introduce slack variables S_1, S_2, S_3

$$\Longrightarrow \quad 3x - y + 2z + S_1 = 7$$
$$-2x - 4y + S_2 = 12$$
$$-4x + 3y + 8z + S_3 = 10$$

			-1	-3	-3	0	0	0	
NZV	C_B	X_B	x	y	z	S_1	S_2	S_3	Ratios
S_1	0	7	3	-1	2	1	0	0	$7/-1$ (Ignore)
S_2	0	12	-2	-4	0	0	1	0	$12/-4$ (Ignore)
S_3	0	10	-4	$\textcircled{3}$	8	0	0	1	$10/3$ ←
$Z_j - C_j$	$P' = 0$		1	-3 ↑	3	0	0	0	–

Pivot = 3 Apply $R_3' \to R_3/3$

OGV = S_3 $R_1' \to R_1 + R_3'$

ICV = y $R_2' \to R_2 + 4R_3'$

			-1	$+3$	-3	0	0	0	
NZV	C_B	X_B	x	y	z	S_1	S_2	S_3	Ratios
S_1	0	$31/3$	$5/3$	0	$14/3$	1	0	$1/3$	$31/5$ ←
S_2	0	$76/3$	$-22/3$	0	$32/3$	0	1	$4/3$	$76/-22$ (Ignore)
y	3	$10/3$	$-4/3$	1	$8/3$	0	0	$1/3$	$10/-4$ (Ignore)
$Z_j - C_j$	$P' = 10$		-3 ↑	0	11	0	0	1	

Pivot = $5/3$ Apply $R_1' \to 3/5\ R_1$

OGV = S_1 $R_2' \to R_2 + 22/3\ R_1'$

ICV = x $R_3' \to R_3 + 4/3\ R_1'$

NZV	C_B	X_B	x	y	z	S_1	S_2	S_3
x	-1	31/5	1	0	14/5	3/5	0	1/5
S_2	0	354/5	0	0	156/5	22/5	1	14/5
y	3	58/5	0	1	32/5	4/3	0	3/5
$z_j - c_j$	$P' = $ 143/5		0	0	97/5	17/5	0	9/5

$P'_{max} = \frac{143}{5}$ $\Rightarrow$ $P_{min} = -\frac{143}{5}$

at $x = \frac{31}{5}, y = \frac{58}{5}, z = 0$ at $x = \frac{31}{5}, y = \frac{58}{5}, z = 0$

③ A company assembles 3 types in trains, trucks & cars using 3 operations. The daily limits on the available time for the 3 operations are 430, 460 & 420 minutes resp. The assembly times/train at the 3 operations are 1, 3 & 1 minutes resp. The corresponding times per truck & per car are 204, and 120 resp. Profit per train is Rs. 3, profit per 1 truck is Rs. 2 & profit per car is Rs. 5. Formulate the problem as a LP model and solve the LPP by simplex method.

Soln Let x, y, z be the total no of trains, trucks & cars manufactured. The given data is given by following table

limits	train	truck	car
430	1	2	1
460	3	0	2
420	1	4	0
Profit	3	2	5

LP model

Max $P = 3x + 2y + 5z$

STC $x + 2y + z \le 430$

$$3x + 2z \leq 460$$

$$x + 4y \leq 420$$

$$x, y, z \geq 0$$

UNIT 3

SIMPLEX METHOD-2

Post Optimality Analysis

Post optimality analysis - the analysis done after an optimal solution is obtained for the initial version of the model - constitutes a very major and very important role in most of OR studies. The fact that post-optimality analysis is very important is particularly true for typical LP applications.

Typical steps in post-optimality analysis for LP Studies

The following table summarizes typical steps in post optimality analysis. The rightmost column identifies some methods that involve simplex method.

TASK	PURPOSE	TECHNIQUE
Model Debugging	Find errors & weakness in model	**Reoptimization**
Model Validation	Demonstrate validity of final model	Retrospective Test
Final managerial decisions on resource allocations	Make appropriate division of organizational resources between activities under study and other important activities.	Shadow pieces
Evaluate estimates of model parameters	Determine crucial estimates that may affect optimal solution for further study.	Sensitivity analysis
Evaluate trade-offs between model parameters	Determine best trade-off	Parametric Linear Programming

Reoptimization

LP models that arise in practice are very large with thousands or even millions of functional constraints and decision variables. After having found an optimal solution for one version of LP model, we frequently must solve again for the solution of a slightly different version of the model. One approach is re-apply simplex method from scratch for each new-version of the model. However a much more efficient approach is to re-optimize. If the final solution is feasible, we apply usual simplex method and if it ~~is~~ not feasible, then we apply dual simplex method. The big advantage of this re-optimization technique over re-solving from the scratch is that an optimal solution for the revised model probably is going to be much closes to the prior optimal solution than to an initial BFS.

Retrospective Test

A systematic approach to testing the model is called a retrospective test. This test involves using the historical data to reconstruct the past and then determine how well the model and the resulting solution would have performed if they had been used.

Shadow Pieces

LPP often can be interpreted as allocating resources to activities. In many cases, there may be some latitude in the amounts that will be made available.

Information on the economic contribution of the resources to the measure of performance (z) for the current study often would be extremely useful. The simplex method provides this information in the form of shadow pieces for the respective resources.

Sensitivity Analysis

A model is never a perfect representation of reality. But, if properly formulated and correctly manipulated, it may be useful in predicting the effect of changes in control variables on the overall system effectiveness. The usefulness of a model is tested by determining how well it predicts the effect of these changes. Such an analysis is called sensitivity analysis. The utility or validity of the solution can be checked by comparing the results obtained to without applying the solution with the results obtained when it is used.

Parametric Linear Programming

This investigates the effect of pre-determined continuous variations of various co-efficients on the optimal solution. It is extension of sensitivity analysis and aims at finding the various basic solutions that become optimal, one after the other, as the co-efficients of the problem change continuously. The coefficients change as a linear function of a single parameter, hence the name parametric linear programming for this computational technique.

Computer Implementation

Computer codes for the simplex method now are widely available for essentially all modern computer systems. These codes commonly are part of a sophisticated software package for mathematical programming. These production computer codes don't closely follow either the algebraic form or the tabular form of simplex method. These forms can be streamlined considerably for computer implementation. Therefore the codes use instead a matrix form (usually called the revised simplex method) that is especially well suited for the computer. This form accomplishes exactly the same things as the algebraic or tabular form, but it does this while computing and storing only the numbers that are actually needed for the current iteration; and then it carries along the essential data in a more compact form.

With large LPP, it is inevitable that some mistakes and faulty decisions will be made initially in formulating the model and inputting it into a computer. OR typically consider a long series of variations on a basic model to examine different scenarios as part of post-optimality analysis. This entire process is greatly accelerated when it can be carried out interactively on a computer.

Available software options for Linear Programming

1. Excel and its premium solver for formulating and solving LP models on a spreadsheet format : Spreadsheet solvers are becoming increasingly popular for linear programming. Leading the way are the solvers produced by Frontline systems for Microsoft Excel and other spreadsheet packages.

2. MPL / CPLEX for efficiently formulating and solving large linear programming models: one that is widely regarded to be a particularly powerful package for solving massive problems is CPLEX a product of ILOG. CPLEX often uses the simplex method and its variants to solve these massive problems.

3. LINGO and its solver (shared with LINDO) for an alternative way of efficiently formulating and solving large linear programming models. LINDO [short for LINEAR, INTERACTIVE. AND DISCRETE OPTIMIZE] is a software which solves LP model in a straightforward way.

UNIT 3

ARTIFICIAL VARIABLE TECHNIQUES

LPPs in which constraints of the type ≥ and/or = are present surplus variables and artificial variables are introduced.

TWO - PHASE METHOD

Working Rule: PHASE - I

Step 1 : Convert the given OF to maximization type (if required)

Step 2 : Convert each inequality of the type ≤ to = by introducing slack variables. Convert each inequality of the type ≥ to = by introducing surplus variables and artificial variables. If the constraint is of the type =, then an artificial variable is introduced. Thus, the given LPP is converted into standard form

Step 3 : Assign a cost -1 to each AV and a cost 0 to all other variables.

Step 4 : Construct the auxillary LPP in which the new OF Z^* is to be maximized.

Step 5 : Solve the auxillary LPP by simplex method until one of the following possibilities arise:

(i) Maximum $Z^* = 0$ and atleast one AV appears in the optimum basis at 0 level. In this case, LPP has a solution & to find the solution, go to Phase-2

(ii) Maximum $Z^* = 0$ and no AV appears in the optimum basis. In this case, problem has a solution and to find the solution, go to Phase - 2

(iii) Maximum $Z^* = -ve$ & atleast one AV appears in the optimum basis at a positive level. In this case, the given LPP doesnot possess feasible solution

Working Rule : PHASE-II

Consider the final simplex table obtained at the end of Phase-I. Assign the actual cost in the OF and a 0 cost to every AV that appears in the optimum basis at 0 level. The new OF is maximized by simplex method.

<u>NOTE</u> : Whenever an AV becomes an outgoing variable, we drop the column corresponding to the AV in subsequent steps.

1. Minimize $P = 7.5x - 3y$

STC $3x - y - z \geq 3$

$x - y + z \geq 2$

$x, y, z \geq 0$

Solution : We know maximization = −(minimization).

$\therefore$ Max $P' = -7.5x + 3y$

Introduce two surplus variables

$3x - y - z - S_1 = 3$

$x - y + z - S_2 = 2$

Put $x = y = z = 0$ (set design)

$-S_1 = 3$ Not feasible

$-S_2 = 2$ Not feasible

$\therefore$ We introduce two AVs a_1 and a_2

$3x - y - z - S_1 + a_1 = 3$

$x - y + z - S_2 + a_2 = 2$

$\therefore a_1 = 3$ & $a_2 = 2$

Phase-I

Assign a cost −1 to AVs a_1 and a_2 and a cost '0' to all other variables.

			0	0	0	0	0	-1	-1	
NZV	C_B	X_B	x	y	z	S_1	S_2	a_1	a_2	Ratios
a_1	-1	3	$\textcircled{3}$	-1	-1	-1	0	1	0	1 ←
a_2	-1	2	1	-1	+1	0	-1	0	1	2
$z_j - c_j$	$P^* = -5$		-4	2	0	1	1	0	0	
			↑							

Pivotal element = 3 Apply $R_1' \to R_1/3$

OGV = a_1 $R_2' \to R_2 - R_1'$

ICV = x

As OGV is an AV, we shall drop the column corresponding to a_1 in the next simplex table

			0	0	0	0	0	-1	
NZV	C_B	X_B	x	y	z	S_1	S_2	a_2	Ratios
x	0	1	1	-1/3	-1/3	-1/3	0	0	IGNORE
a_2	-1	1	0	-2/3	$\textcircled{4/3}$	1/3	-1	1	3/4 ←
$z_j - c_j$	$P^* = -1$		0	2/3	-4/3	-1/3	1	0	
					↑				

Pivot = 4/3 Apply $R_2' \to 3/4\, R_2$

OGV = a_2 $R_1' \to R_1 + 1/3\, R_2'$

ICV = z

As the OGV is an AV, we shall drop the column corresponding to a_2 in the next simplex ~~table~~.

			0	0	0	0	0
NZV	C_B	X_B	x	y	z	S_1	S_2
x	0	5/4	1	-1/2	0	-1/4	-1/4
z	0	3/4	0	-1/2	1	1/4	-3/4
$z_j - c_j$	$P^* = 0$		0	0	0	0	0

~~Plea~~ We note that $\overset{max}{\uparrow} P^* = 0$ and no AV appears in the optimum basis. ∴ The given LPP has a solution.

Phase-II :

Consider the final simplex table obtained at the end of phase I. Assign the actual cost of the OF

			-7.5	3	0	0	0
NZV	CB	XB	x	y	z	s_1	s_2
x	-7.5	5/4	1	-1/2	0	-1/4	-1/4
z	0	3/4	0	-1/2	1	1/4	-3/4
$z_j - c_j$	$P' = -75/8$		0	3/4	0	15/8	15/8

As all the indicators are non negative, simplex procedure is complete. $\boxed{P'_{max} = -\frac{75}{8}}$ at $x = \frac{5}{4}$, $y = 0$, $z = \frac{3}{4}$

$\therefore$ $\boxed{P_{min} = +\frac{75}{8}}$ at $x = \frac{5}{4}$, $y = 0$, $z = \frac{3}{4}$

- Maximize $z = 5x + 3y$

STC $2x + y \leq 1$

$x + 4y \geq 6$

$x, y \geq 0$

Solution: Introduce a slack variable s_1

surplus s_2

$2x + y + s_1 = 1$

$x + 4y - s_2 = 6$

Put $x = 0$, $y = 0$

$s_1 = 1$ feasible

$-s_2 = 6$

$s_2 = -6$ not feasible

Introduce a one av a_1

$2x + y + s_1 = 1$

$x + 4y - s_2 + a_1 = 6$

Phase-I : Assign a cost -1 to AV a_1 and cost 0 to all other variables

NZV	C_B	X_B	x	y	S_1	S_2	a_1	Ratios
s_1	0	1	2	①	1	0	0	$1/1 = 1$ ←
a_1	-1	6	1	4	0	-1	1	$6/4 = 1.5$
$z_j - c_j$	$z^* = -6$		-1	-4 ↑	0	1	0	

OGV = S_1 Apply $R_2' \to R_2 - 4R_1$

ICV = y

NZV	C_B	X_B	x	y	S_1	S_2	a_1
y	0	1	2	1	1	0	0
a_1	-1	2	-7	0	-4	-1	1
$z_j - c_j$	$z^* = -2$		7	0	4	1	0

~~The~~ ~~a~~ As max z^* is negative, & an AV appears at a positive level, the given LPP doesnot possess a solution.

Dec 2011

Food X contains 6 units of Vitamin A / gram & 7 units of Vitamin B / gram. The cost of food X is 12ps / gm

Food Y contains 8 units of vitamin A / gm & 12 units of Vitamin B / gm. The cost of food Y is 20ps / gm

The daily minimum requirements of vitamin A & vitamin B are 100 units & 120 units respectively. Find the minimum cost of the product mix. Formulate the problem as a LP model and solve the resulting LPP by two-phase method.

Solution

Given data is put in tabular form as shown:

	Vit A	Vit B	Cost
Food X	6	7	12 paise/gm
Food Y	8	12	20 paise/gm
Daily min requirements	100	120	

Let x be the total no of units of food X

y $\qquad$ y

Given: cost/gram of food X is 12 paise

$\Rightarrow$ cost/~~gm~~ x grams of food X is $12x$

Given: cost/gram of food Y is 20 paise

$\Rightarrow$ cost/y grams of food Y is $20y$

$\therefore$ Total cost $= \underline{12x + 20y}$

The OF is Min $z = 12x + 20y$

Food X & Food Y cannot be negative $\Rightarrow x, y \geq 0$

Food X contains 6 units of Vit A and food Y contain 8 units of Vit A

$\Rightarrow$ total Vit A $= 6x + 8y$

But a min of 100 units of Vit A are required

$\Rightarrow 6x + 8y \geq 100$

Food X contains 7 units of Vit B & food Y contains 12 units of Vit B

$\Rightarrow$ total vit B $= 7x + 12y$

But a min of 120 units of Vit B are required.

$\Rightarrow 7x + 12y \geq 120$

$\therefore$ Given LPP is

$$\text{Min } z = 12x + 20y$$
$$\text{STC } \quad 6x + 8y \geq 100$$
$$7x + 12y \geq 120$$
$$x, y \geq 0$$

$$\text{Max } z' = -12x - 12y$$

Introduce two surplus variables & hence 2 AV.

$$6x + 8y - S_1 + a_1 = 100$$
$$7x + 12y - S_2 + a_2 = 120$$

Phase-I : Assign a cost -1 to each A.V. & a cost 0 to all other variables.

			0	0	0	0	-1	-1	
NZV	C_B	X_B	x	y	S_1	S_2	a_1	a_2	Ratios
a_1	-1	100	6	8	-1	0	1	0	$100/8 = 12.5$
a_2	-1	120	7	$\boxed{12}$	0	-1	0	1	$120/12 = 10$ ←
$z_j - c_j$	$Z^* = -220$		-13	-20	1	1	0	0	—
				↑					

OGV = a_2 Apply $R_2' \to R_2/12$

ICV = y $R_1' \to R_1 - 8R_2'$

Pivot = 12

As OGV is an AV, we shall drop the column corresponding to a_2 in the ~~si~~ next simplex table.

			0	0	0	0	-1	
NZV	CB	X_B	x	y	S_1	S_2	a_1	Ratios
a_1	-1	20	4/3	0	-1	2/3	1	15 ←
y	0	10	7/12	1	0	-1/12	0	120/7
$z_j - c_j$	$Z^* = -20$		-4/3	0	1	-2/3	0	—
			↑					

OGV = a_1 Apply $R_1' \to \frac{3}{4} R_1$

ICV = x $R_2' \to R_2 - \frac{7}{12} R_1'$

As OGV is an AV, we shall drop the column corresponding to a_1 in the next simplex table

			0	0	0	0
NZV	C_B	X_B	x	y	S_1	S_2
x	0	15	1	0	$-3/4$	$1/2$
y	0	$5/4$	0	1	$7/16$	$-3/8$
$z_j - c_j$	$z^* = 0$		0	0	0	0

As max $z^* = 0$ & as all indicators are non-negative, phase-I is complete. We also note that AVs donot appear in the optimum basis.

Phase II

Start with the final table of phase I. Assign the actual cost to the variables as given in OF.

NZV	C_B	X_B	x	y	S_1	S_2
x	-12	15	1	0	$-3/4$	$1/2$
y	-20	$5/4$	0	1	$7/16$	$-3/8$
$z_j - c_j$	$Z' = -205$		0	0	$1/4$	$3/2$

As all the indicators are non-negative, simplex procedure is complete. $\therefore Z'_{max} = -205$ & occurs at $x = 15$ & $y = \frac{5}{4}$

$\Rightarrow Z_{min} = 205$ at $x = 15$ & $y = \frac{5}{4}$

A/W Dec 2010

Use two-phase method to minimize $z = 0.4x + 0.5y$

$$\text{STC } 0.3x + 0.1y \le 2.7$$
$$0.5x + 0.5y = 6$$
$$0.6x + 0.4y \ge 6$$

Soln. We min = -(max)

Therefore max $z' = -0.4x - 0.5y$

The first constraint is of type $\leq$, $\therefore$ we introduce a slack variable s_1. $\Rightarrow 0.3x + 0.1y + s_1 = 2.7$

The second constraint is of type $=$, $\therefore$ we introduce an AV, a_1. $\Rightarrow 0.5x + 0.5y + a_1 = 6$

We note that third constraint is of type $\geq$, $\therefore$ we introduce a surplus variable s_2 & av a_2.

$$\therefore \; 0.6x + 0.4y - s_2 + a_2 = 6$$

SOLVE LATER

HW

4 Using 2 phase method, maximize $\phi = 5x - 4y + 3z$

$$\text{STC.} \quad 2x + y - 6z = 20$$
$$6x + 5y + 10z \leq 76$$
$$8x - 3y + 6z \leq 50$$
$$x, y, z \geq 0$$

Charne's Penalty Method (Big-M Method)

Working Rule

Step 1 : Convert the OF to maximization type if required

Step 2 : Convert the given linear inequalities into equalities by introducing slack variables or surplus variables & a.v.

NOTE : If a given constraint is of ~~eq~~ the type '=' an a.v. is introduced on the left hand side.

Step 3 : Rewrite the OF as $\text{Max } P = a_1x + b_1y + c_1z + \dots - MA_1 - MA_2 \dots$ where M is called as Penalty. This penalty is introduced to avoid a.v.s in the optimum basis at a positive level.

Step 4 : Solve the modified LPP by Simplex method & we stop the procedure when one of the following 3 cases arise :-

Case (i) : If no a.v appears in the optimum basis & all the indicators are non-negative, then the current soln is the optimal soln.

Case (ii) : If all the Indicators are nonnegative & atleast one a.v appears in the optimum basis at zero level, then the current soln is the optimal soln.

Case (iii) : If the optimality condition is satisfied, and atleast one a.v appears in the optimum basis at a positive level, then the given LPP has no feasible solution or LPP has a pseudooptimal solution.

NOTE : Whenever an a.v. becomes an OGV, we shall ignore the column corresponding to the a.v. in the subsequent simplex tables.

Using Penalty method, minimize $z = 4x + y$

STC $3x + y = 3$

$4x + 3y \geq 6$

$x + 2y \leq 4$

$x, y \geq 0$

Solution : As the first constraint is of the type '=', we introduce an a.v a_1

$$3x + y + a_1 = 3$$

As second constraint is of the type '≥', we introduce a surplus variable S_1 and hence an a.v a_2.

$$4x + 3y - S_1 + a_2 = 6$$

As the third constraint is of the type '≤', we introduce a slack variable S_2 i.e.

$$x + 2y + S_2 = 4$$

We know minimization = -(maximization)

$$\text{Max } Z' = -4x - y$$

as two a.v.s a_1 and a_2 are introduced, the new OF is

$$\text{Max } Z' = -4x - y - Ma_1 - Ma_2$$

			-4	-1	0	0	-M	-M	
NZV	C_B	X_B	x	y	S_1	S_2	a_1	a_2	Ratios
a_1	-M	3	③	1	0	0	1	0	3/3 = 1 ←
a_2	-M	[illegible]	4	3	-1	0	0	1	6/4 = 1.5
S_2	0	4	1	2	0	1	0	0	4/1 = 4
$Z_j - C_j$	$Z' = -9M$		-7M+4 ↑	-4M+1	M	0	0	0	-

Pivot = 3 Apply $R_1' \to R_1/3$

OGV = a_1 $R_2' \to R_2 - 4R_1'$

ICV = x $R_3' \to R_3 - R_1'$

As the OGV is an a.v., we shall drop the column corresponding to a_1 in the next simplex table

			-4	-1	0	0	-M	
NZV	C_B	X_B	x	y	S_1	S_2	a_2	Ratios
x	-4	1	1	1/3	0	0	0	$1/_{1/3} = 3$
a_2	-M	2	0	$\boxed{5/3}$	-1	0	1	$2/_{5/3} = 6/5$ ←
S_2	0	3	0	5/3	0	~~0~~ 1	0	9/5
Ind	$Z' = -4-2M$		0	$\frac{-5M}{3} - \frac{1}{3}$ ↑	M	0	0	

Pivot = 5/3

OGV = a_2

ICV = y

Apply $R_2' \rightarrow R_2 \times 3/5$

$R_1' \rightarrow R_1 - 1/3 R_2'$

$R_3' \rightarrow R_3 - 5/3 R_2'$

as OGV is an a.v, we can drop the column corresponding to a_2 in the next simplex table

			-4	-1	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	Ratios
x	-4	3/5	1	0	1/5	0	3
y	-1	6/5	0	1	-3/5	0	IGNORE
S_2	0	1	0	0	$\boxed{1}$	1	1 ←
$Z_j - C_j$	$Z' = -18/5$		0	0	-1/5 ↑	0	

Pivot = 1

OGV = S_2

ICV = S_1

Apply $R_1' \rightarrow R_1 - 1/5 R_3$

$R_2' \rightarrow R_2 + 3/5 R_3$

NZV	C_B	X_B	x	y	S_1	S_2
x	-4	2/5	1	0	0	-1/5
y	-1	9/5	0	1	0	3/5
S_1	0	1	0	0	1	1
Ind	$Z' = -17/5$		0	0	0	1/5

As all indicators are non-negative, simplex method is complete.

$Z'_{max} = \frac{-17}{5}$ at $x = \frac{2}{5}$ & $y = \frac{9}{5}$

$\Rightarrow Z_{min} = \frac{17}{5}$ at $x = \frac{2}{5}$ & $y = \frac{9}{5}$

Using Big-M method, maximize $z = 2x + y$

July 2009
Dec 2010

STC $3x + y = 3$

$4x + 3y \geq 6$

$x + 2y \leq 3$

$x, y \geq 0$

Solution : The first constraint is of the type '=', therefore, we introduce an av a_1.

$\Rightarrow 3x + y + a_1 = 3$

The second constraint is of the type '≥', therefore, we introduce a surplus variable S_1 & hence an av. a_2

$\Rightarrow 4x + 3y - S_1 + a_2 = 6$

As the 3rd constraint is of the type '≤', we introduce a slack variable S_2.

$\Rightarrow x + 2y + S_2 = 3$

As two a.v.s are introduced, the new OF is

$\Rightarrow$ Max $Z = 2x + y - Ma_1 - Ma_2$

			2	1	0	0	-M	-M	
NZV	C_B	X_B	x	y	S_1	S_2	a_1	a_2	Ratios
a_1	-M	3	$\textcircled{3}$	1	0	0	1	0	$3/3 = 1$ ←
a_2	-M	6	4	3	-1	0	0	1	$6/4 = 1.5$
S_2	0	3	1	2	0	1	0	0	$3/1 = 3$
$Z_j - C_j$		$z = -9M$	-7M-2	-4M-1	M	0	0	0	
			↑						

Pivot = 3

OGV = a_1

ICV = x

Apply $R_1' \rightarrow R_1/3$

$R_2' \rightarrow R_2 - 4R_1'$

$R_3' \rightarrow R_3 - R_1'$

			2	1	0	0	-M	
NGV	C_B	X_B	x	y	S_1	S_2	a_2	Ratios
x	2	1	1	1/3	0	0	0	3
a_2	-M	2	0	5/3	-1	0	1	6/5
S_2	0	2	0	$\boxed{5/3}$	0	1	0	6/5 ←
$Z_j - C_j$	$Z = -2M+2$		0	$-\frac{5M}{3} - \frac{1}{3}$	M	0	0	
				↑				

Max $z = 3x + 2y$

STC $2x + y \leq 2$

$3x + 4y \geq 12$

$x, y \geq 0$

			3	2	0	0	-M	
NZV	C_B	X_B	x	y	S_1	S_2	a_1	Ratios
S_1	1	2	2	$\boxed{1}$	1	0	0	2 ←
a_1	-M	12	3	4	0	-1	1	3
$Z_j - C_j$		$Z = -12M$	-3M-3	-4M-2 ↑	0	M	0	-

OGV = S_1 Apply $R_2' \to R_2 - 4R_1$

ICV = y

NZV	C_B	X_B	x	y	S_1	S_2	a_1
y	2	2	2	1	1	0	0
a_1	-M	4	-5	0	-4	-1	1
$Z_j - C_j$		$Z = 4 - 4M$	5M+1	0	+4M+2	M	0

As all indicators are non negative, simplex procedure is complete. As a.v. a_1 appears in the optimum basis at a positive level, the given LPP doesnot have a solution.

Using Big-M method, Max $z = 6x + 4y$

STC $2x + 3y \leq 30$

$3x + 2y \leq 24$

Dec 2011

$x + y \geq 3$

$x, y \geq 0$

Find atleast two solutions

Solution: The first two constraints are of the type $\leq$ ∴ we introduce two slack variables S_1 & S_2

Third constraint is of the type $\geq$ ∴ we introduce a surplus variable S_3 & an a.v. a_1.

$$2x + 3y + S_1 = 30$$

$$3x + 2y + S_2 = 24$$

$$x + y - S_3 + a_1 = 3$$

The OF is $6x + 4y - Ma_1$

			6	4	0	0	0	−M	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	a_1	Ratios
S_1	0	30	2	3	1	0	0	0	15
S_2	0	24	3	2	0	1	0	0	8
a_1	−M	3	①	1	0	0	−1	1	3 ←
$Z_j - C_j$	Z = −3M	−M−6	−M−4	0	0	M	0	−	
			↑						

We note that there is a tie between 1st & 2nd column
We shall select the first column as the pivotal column

OGV = a_1 Apply $R_1' \rightarrow R_1 - 2R_3$

ICV = x $R_2' \rightarrow R_2 - 3R_3$

			6	4	0	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	Ratios
S_1	0	24	0	1	1	0	2	12
S_2	0	15	0	−1	0	1	③	5 ←
x	6	3	1	1	0	0	−1	IGNORE
Ind	z = 18	0	2	0	0	−6	−	
							↑	

Pivot = 3, OGV = S_2, ICV = S_3

Apply $R_2' \to R_2/3$, $R_1'' \to R_1 - 2R_2'$, $R_3' \to R_3 + R_2'$

			6	4	0	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	~~Ratios~~
S_1	0	14	0	5/3	1	-2/3	0	
S_3	0	5	0	-1/3	0	1/3	1	
x	6	8	1	2/3	0	1/3	0	
Ind	$z = 48$		0	0	0	2	0	

As all indicators are non-negative, simplex procedure is complete.
$Z_{max} = 48$ and it occurs at $x = 8$ and $y = 0$.

~~Case (ii)~~ ALTERNATE SOLUTION (select 2nd column as pivotal column)

			6	4	0	0	0	-M	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	a_1	Ratios
S_1	0	30	2	3	1	0	0	0	15
S_2	0	24	3	2	0	1	0	0	8
a_1	-M	3	1	$\boxed{1}$	0	0	-1	1	3 ←
$Z_j - C_j$	$Z = -3M$		-M-6	-M-4 ↑	0	0	M	0	-

OGV = a_1 Apply $R_1' \to R_1 - 3R_3$

ICV = y $R_2' \to R_2 - 2R_3$

			6	4	0	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	Ratios
S_1	0	21	-1	0	1	0	$\boxed{3}$	7 ←
S_2	0	18	1	0	0	1	+2	9 ~~Ignore~~
y	4	3	1	1	0	0	-1	IGNORE
Ind	$Z = 12$		-2	0	0	0	-4 ↑	-

OGV = S_1 Apply $R_1' \to R_1/3$

ICV = S_3 $R_2' \to R_2 - 2R_1'$

$R_3' \to R_3 + R_1'$

			6	4	0	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	Ratios
S_3	0	7	$-1/3$	0	$1/3$	0	1	Ignore
S_2	0	4	$\boxed{5/3}$	0	$-2/3$	1	0	$12/5$ ←
y	4	10	$2/3$	1	$1/3$	0	0	15
Ind	$Z = 40$		$-10/3$ ↑	0	$4/3$	0	0	-

OGV = S_2

ICV = x

Apply $R_2' \to 3/5\, R_2$

$R_1' \to R_1 + 1/3\, R_2'$

$R_3' \to R_3 - 2/3\, R_2'$

			6	4	0	0	0	
NZV	C_B	X_B	x	y	S_1	S_2	S_3	~~Ratios~~
S_1	0	$39/5$	0	0	$1/5$	$1/5$	1	
x	6	$12/5$	1	0	$-2/5$	$3/5$	0	
y	4	$42/5$	0	1	$3/5$	$-2/5$	0	
Ind	$Z = 240/5 = 48$		0	0	0	2	0	

$Z_{max} = 48$ at $x = \frac{12}{5}$ and $y = \frac{42}{5}$

AW Using Big-M method, minimize $P = 3x + 2y + z$

Jan 2010 (12M)

STC $x + y = 7$

$3x + y + z \geq 10$

$x, y, z \geq 0$

Use Penalty Method to minimize $z = 5x + 2y$

July 2011 (16M)

STC $2x + 4y \leq 12$

$2x + 2y = 10$

$5x + 2y \geq 10$

$x, y \geq 0$

GAME THEORY & DECISION ANALYSIS

Game:

Game is defined as an activity between 2 or more persons involving activities by each person according to a set of rules at the end of which each person receives some benefit or suffers loss. If in a game, activities are determined by skill, it is called as game of strategy. If the activities are determined by probability, then the game is called as a game of chance.

2 persons: Zero Sum

A game with two persons is called as a 2 persons: Zero Sum game if the losses of one player is equal to the gains of other player so that the net sum of their gains is zero

Pay-off matrix

Suppose a player A has m-activities and player B has n-activities, m need not be equal to n. Then a pay-off matrix is constructed using the following conventions

- Row designations of the matrix are the activities available to player A.
- Column designations are the activities available to player B.
- Cell entry V_{ij} is the payment made by player B to player A when player A selects strategy i & player B selects strategy j.

$$\begin{array}{cc|ccccc|} & & & & Ⓑ & & \\ & & 1 & 2 & 3 & \cdots & n \\ \hline & 1 & V_{11} & V_{12} & V_{13} & \cdots & V_{1n} \\ & 2 & V_{21} & V_{22} & V_{23} & \cdots & V_{2n} \\ Ⓐ & 3 & V_{31} & V_{32} & V_{33} & \cdots & V_{3n} \\ & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & m & V_{m1} & V_{m2} & V_{m3} & \cdots & V_{mn} \\ \hline \end{array}$$

UNIT 6

Divya

A transportation problem is to transport a single homogeneous commodity that are initially stored at 'm' origins to 'n' different destinations such that the total transportation cost is minimum.

General Transportation Table

Dest^n origin	1	2	3	......	n	Supply
1	C_{11}	C_{12}	C_{13}	......	C_{1n}	a_1
2	C_{21}	C_{22}	C_{23}	......	C_{2n}	a_2
3	C_{31}	C_{32}	C_{33}	......	C_{3n}	a_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮
m	C_{m1}	C_{m2}	C_{m3}	-----	C_{mn}	a_m
Demand	b_1	b_2	b_3	......	b_n	

* This table represents a m-origins and n-destinations transportation problem (m need not be equal to n)

* If $\Sigma a_i = \Sigma b_j$, then it is called as a balanced transportation problem; otherwise it is called as an unbalanced transportation problem.

* C_{ij} is the amount of money spent in transporting one commodity from i^{th} origin to j^{th} destination.

Feasible Solution of Transportation Problem

A set of positive allocations which removes deficiencies is called as a feasible solution

Basic Feasible Solution (BFS)

A feasible solution is called as a Basic Feasible solution if total number of allocations $= m+n-1$ i.e. one less than the total number of rows and columns of the transportation table.

If the total allocations is exactly $m+n-1$, then the solution is also called as non-degenerate basic feasible solution.

If the total allocations is less than $m+n-1$, then it is called as degenerate basic feasible solution.

Optimal Solution

A feasible solution (not necessarily basic feasible) is called the optimal solution if it minimizes the total transportation Cost

METHODS TO FIND INITIAL SOLUTION

1. North West Corner Rule (NWC)
2. Matrix - Minima Method
3. Vogel's Approximation Method (VAM)

NORTH-WEST CORNER RULE (NWC)

Working Rule:

Step 1: Identify the North-West corner of the table (In the first step, the NWC will be the cell (1,1)).

Allocate $x_{11} = \min(a_1, b_1)$

Case 1: If $a_1 < b_1$, then first row gets completed.

Case 2: If $b_1 < a_1$, then first column gets completed.

Case 3: If $a_1 = b_1$, then there is a tie and allocation can be made arbitrarily.

Step 2: Start from the new north-west corner and repeat Step 1 until all the requirements are satisfied.

PROBLEMS

① Find an initial solution for the following transportation problem by NWC method.

	D_1	D_2	D_3	D_4	Supply
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
Demand	5	8	7	14	

Solution:

NOTE: ① Costs are written at the bottom-left corner

② Allocations are written at the centre of the box.

Supply = 7+9+18 = 34

Demand = 5+8+7+14 = 34

As supply = demand, the given transportation problem is a balanced transportation problem.

	D_1	D_2	D_3	D_4	Supply
O_1	•5 19	•2 30	50	10	~~7~~ ~~2~~ 0
O_2	70	•6 30	•3 40	60	~~9~~ ~~3~~ 0
O_3	40	8	•4 70	•14. 20	~~18~~ ~~14~~ 0
Demand	~~5~~ 0	~~8~~ ~~6~~ 0	~~7~~ ~~4~~ 0	~~14~~ 0	

Step 1: NWC is (1,1). $x_{11} = \min(7,5) = 5$. C_1-completes

Step 2: NWC is (1,2). $x_{12} = \min(2,8) = 2$. R_1-completes

Step 3: NWC is (2,2). $x_{22} = \min(9,6) = 6$. C_2-completes

Step 4: NWC is (2,3). $x_{23} = \min(3,7) = 3$. R_2-completes

Step 5: NWC is (3,3). $x_{33} = \min(18,4) = 4$. C_3-completes

Step 6: Allocate 14 to the cell (3,4).

We note that total allocations $= 6 = 3+4-1$

$\therefore$ The solution is basic feasible.

Transportation Cost $= (5\times19)+(2*30)+(6\times30)+(3\times40)+(4\times70)+(14\times20)$

$=$ Rs. 1015

---

② Find an initial solution by NWC method

	D_1	D_2	D_3	Supply
O_1	5	7	8	70
O_2	4	4	6	30
O_3	6	7	7	50
Demand	65	42	43	

Solution

Supply $= 70+30+50 = 150$

Demand $= 65+42+43 = 150$

As supply = demand, the TP is balanced

	D_1	D_2	D_3	Supply
O_1	•65 5	•5 7	8	~~70~~ ~~5~~ 0
O_2	4	•30 4	6	~~30~~ 0
O_3	6	•7 7	•43 7	~~50~~ ~~43~~ 0
Demand	~~65~~ 0	~~42~~ ~~37~~ ~~7~~ 0	~~43~~ 0	

Step 1: NWC is (1,1). $x_{11} = \min(70, 65) = 65$. C_1-completes

Step 2: NWC is (1,2). $x_{12} = \min(5, 42) = 5$. R_1-completes

Step 3: NWC is (2,2). $x_{22} = \min(30, 37) = 30$. R_2-completes

Step 4: NWC is (3,2). $x_{32} = \min(7, 50) = 7$. C_2-completes

Step 5: Allocate 43 to cell (3,3)

Total no. of allocations $= 5 = 3 + 3 - 1$

$\therefore$ The solution is initial basic feasible

Cost $= (65 \times 5) + (5 \times 7) + (30 \times 4) + (7 \times 7) + 43 \times 7 = \underline{\underline{810}}$

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③ Find an initial solution by NWC method

	D_1	D_2	D_3		Supply
O_1	2	7	4		5
O_2	3	3	1		8
O_3	5	4	7		7
O_4	1	6	2		14
Demand	7	9	18		

Solution

Supply $= 5 + 8 + 7 + 4 = 34$

Demand $= 7 + 9 + 18 = 34$

$\Rightarrow$ TP is balanced since supply = demand

	D_1	D_2	D_3	Supply
O_1	•5 2	7	4	~~5~~ 0
O_2	•2 3	•6 3	1	~~8~~ ~~6~~ 0
O_3	5	•3 4	•4 7	~~7~~ ~~4~~ 0
O_4	1	6	•14 2	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	

Step 1 : NWC is (1, 1) . $x_{11} = \min(5,7) = 5$. R_1-completes

Step 2 : NWC is (2, 1). $x_{21} = \min(8,2) = 2$. C_1-completes

Step 3 : NWC is (2, 2). $x_{22} = \min(6,9) = 6$. R_2-completes

Step 4 : NWC is (3, 2) . $x_{32} = \min(7,3) = 3$. C_2-completes

Step 5 : NWC is (3, 3). $x_{33} = \min(4,18) = 4$. R_3-completes

Step 6 : Allocate 14 to cell (4, 3).

Total no. of allocations = 6 = 4+3-1

∴ The solution is initial basic feasible

Cost = $(5\times 2) + (2\times 3) + (6\times 3) + (3\times 4) + (4\times 7) + (14\times 2)$

= 102

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④ Find an initial basic feasible solution by NWC

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	6
O_3	4	3	6	2	3
Demand	6	10	15	4	

Solution :

Supply = 14 + 6 + 3 = 23

Demand = 6 + 10 + 15 + 4 = 35

As supply ≠ demand ⟹ TP is unbalanced

Dummy row = 35 - 23 = 12

	D1	D2	D3	D4	Supply
O1	•6 6	•8 4	1	5	~~14~~ ~~8~~ 0
O2	8	•2 9	•4 2	7	~~6~~ ~~4~~ 0
O3	4	3	•3 6	2	~~3~~ 0
O4	0	0	•8 0	•4 0	~~12~~ ~~4~~ 0
Demand	~~6~~ 0	~~10~~ ~~2~~ 0	~~15~~ ~~11~~ ~~8~~ 0	~~4~~ 0	

Step 1: NWC is (1,1). $x_{11} = \min(14,6) = 6$. C_1-completes

Step 2: NWC is (1,2). $x_{12} = \min(8,10) = 8$. R_1-completes

Step 3: NWC is (2,2). $x_{22} = \min(6,2) = 2$. R_2-completes

Step 4: NWC is (2,3). $x_{23} = \min(4,15) = 4$. R_2-completes

Step 5: NWC is (3,3). $x_{33} = \min(3,11) = 3$. R_3-completes

Step 6: NWC is (4,3). $x_{43} = \min(12,8) = 8$. C_3-completes

Step 7: Allocate 4 to (4,4).

Total no of allocations = 7 = 4 + 4 − 1

Cost $= (6\times6) + (4\times8) + (9\times2) + (2\times4) + (4\times3) + 0 + 0 = \underline{\underline{112}}$

---

⑤ Find an initial basic feasible solution by NWC

	D1	D2	D3	D4	Supply
O1	13	11	15	20	2000
O2	17	14	12	13	6000
O3	18	18	15	12	7000
	▨	▨	▨	▨	
Demand	3000	3000	4000	5000	

Solution:

Supply = 2000 + 6000 + 7000 = 15000

Demand = 3000 + 3000 + 4000 + 5000 = 15000

⟹ TP is balanced.

	D_1	D_2	D_3	D_4	Supply
O_1	•2000 13	11	15	20	~~2000~~ 0
O_2	•1000 17	•3000 14	•2000 12	13	~~6000~~ ~~5000~~ ~~2000~~ 0
O_3	18	18	•2000 15	•5000 12	~~7000~~ ~~5000~~ 0
Demand	~~3000~~ ~~1000~~ 0	3000 0	~~4000~~ ~~2000~~ 0	~~5000~~ 0	

Step 1 : NWC is (1,1) . $x_{11} = \min(2000, 3000) = 2000$. R_1-completes

Step 2 : NWC is (2,1) . $x_{21} = \min(6000, 1000) = 1000$. C_1-completes

Step 3 : NWC is (2,2) . $x_{22} = \min(5000, 3000) = 3000$. C_2-completes

Step 4 : NWC is (2,3) . $x_{23} = \min(2000, 4000) = 2000$. R_2-completes

Step 5 : NWC is (3,3) . $x_{33} = \min(7000, 2000) = 2000$. C_3-completes

Step 6 : Allocate 5000 to (3,4)

Total no of allocations = 6 = 3+4-1

Cost $= (13 \times 2000) + (17 \times 1000) + (14 \times 3000) + (12 \times 2000) + (15 \times 2000) + (12 \times 5000)$

$= \underline{1,99,000}$

---

⑥ Find an initial solution of the followings by NWC

	D_1	D_2	D_3	D_4	Supply
O_1	4	6	8	13	50
O_2	13	11	10	8	70
O_3	14	4	10	13	30
O_4	9	10	13	8	50
Demand	25	35	105	20	

Solution

Supply = 50 + 70 + 30 + 50 = 200

Demand = 25 + 35 + 105 + 20 = 185

As supply ≠ demand, ⇒ TP is unbalanced

We introduce a dummy column with demand = 15

All the cost of the dummy column are zeros.

	D1	D2	D3	D4	D5	Supply
O1	•25 4	•25 6	8	13	0	~~50~~ ~~25~~ 0
O2	13	•10 11	•60 10	8	0	~~70~~ ~~60~~ 0
O3	14	4	•30 10	13	0	~~30~~ 0
O4	9	11	•15 13	•20 8	•15 0	~~50~~ ~~35~~ ~~15~~ 0
Demand	~~25~~ 0	~~35~~ ~~10~~ 0	~~105~~ ~~45~~ ~~15~~ 0	~~20~~ 0	~~15~~ 0	

Step 1: NWC is (1,1). $x_{11} = \min(50,25) = 25$. C_1 - completes

Step 2: NWC is (1,2). $x_{12} = \min(25,35) = 25$. R_1 - completes

Step 3: NWC is (2,2). $x_{22} = \min(70,10) = 10$. C_2 - completes

Step 4: NWC is (2,3). $x_{23} = \min(60,105) = 60$. R_2 - completes

Step 5: NWC is (3,3). $x_{33} = \min(30,45) = 30$. R_3 - completes

Step 6: NWC is (4,3). $x_{43} = \min(50,15) = 15$. C_3 - completes

Step 7: NWC is (4,4). $x_{44} = \min(35,20) = 20$. C_4 - completes

Step 8: Allocate 15 to x_{45}.

Total no of allocations $= 8 = 4+5-1$

∴ The solution is initial basic feasible.

$$\text{Cost} = (25\times4) + (25\times6) + (11\times10) + (60\times10) + (10\times30) + (13\times15) + (20\times8) + (15\times0)$$

$$= \underline{\underline{1615}}$$

---

⑦ Find an initial solution of the following by NWC

	D1	D2	D3	D4	D5	Supply
O1	4	16	1	16	14	400
O2	18	10	8	12	12	500
O3	6	1	4	13	12	700
Demand	500	400	300	300	600	

Solution:

Supply $= 400 + 500 + 700 = 1600$

Demand $= 500 + 400 + 300 + 300 + 600 = 2100$

Dummy row = 2100 − 1600 = 500

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	400 4	18		16	14	~~400~~ 0
O_2	100 18	400 10	8	12	12	~~500~~ ~~400~~ 0
O_3	6		300 4	300 13	100 12	~~700~~ ~~400~~ ~~100~~ 0
O_4	0	0	0	0	500 0	~~500~~ 0
Demand	~~500~~ ~~100~~ 0	~~400~~ 0	~~300~~ 0	~~300~~ 0	~~600~~ ~~500~~ 0	

Step 1: NWC is (1,1). $x_{11} = \min(400, 500) = 400$. R_1 - completes

Step 2: NWC is (2,1). $x_{21} = \min(500, 100) = 100$. C_1 - completes

Step 3: NWC is (2,2). $x_{22} = \min(400, 400) = 400$. R_2 - completes

Step 4: NWC is (3,2). $x_{32} = \min(700, 0) = 0$. C_2 - completes

Step 5: NWC is (3,3). $x_{33} = \min(700, 300) = 300$. C_3 - completes

Step 6: NWC is (3,4). $x_{34} = \min(400, 300) = 300$. C_4 - completes

Step 7: NWC is (3,5). $x_{35} = \min(100, 600) = 100$. R_4 - completes

Step 8: Allocate 500 to x_{45}.

Total no of allocations = 8 = 4+5−1

∴ The solution is initial basic feasible.

Cost = (400 × 4) + (100 × 18) + (400 × 10) + (300 × 4) + (300 × 13) + (100 × 12)

= 13,700

LEAST COST ENTRY METHOD (MATRIX MINIMA METHOD)

Working Rule:

Step 1: Determine the smallest cost in the transportation table. Let it be c_{ij}. Allocate $x_{ij} = \min(a_i, b_j)$

Step 2: (i) If $x_{ij} = a_i$, then cross-out i^{th} row. Goto step 3.

(ii) If $x_{ij} = b_j$, then cross-out j^{th} column. Goto step 3

(iii) If $x_{ij} = a_i = b_j$, then cross-out i^{th} row or j^{th} column, but not both

Step 3: Repeat steps 1 and 2 for the resulting transportation table until all the requirements are satisfied.

Step 4: Whenever minimum cost is not unique, make an arbitrary choice among the minima.

PROBLEMS

① Solve by Matrix-Minima Method

	D_1	D_2	D_3	D_4	Supply
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
Demand	5	8	7	14	

Solution

Supply = 7 + 9 + 18 = 34

Demand = 5 + 8 + 7 + 14 = 34

⇒ TP is balanced

	D_1	D_2	D_3	D_4	Supply
O_1	19	30	50	•7 10	~~7~~ 0
O_2	•2 70	30	•7 40	60	~~9~~ ~~2~~ 0
O_3	•3 40	•8 8	70	•7 20	18 ~~10~~ ~~3~~ 0
Demand	~~5~~ ~~2~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~7~~ 0	

Step 1: Least cost is 8. $x_{32} = \min(18, 8) = 8$. C_2 - completes

Step 2: Least cost is 10. $x_{14} = \min(7, 14) = 7$. R_1 - completes

Step 3: Least cost is 20. $x_{34} = \min(10, 7) = 7$. C_4 - completes

Step 4: Least cost is 40. There is a tie between (2, 3) & (3, 1). We shall select (2, 3)

$x_{23} = \min(9, 7) = 7$. C_3 - completes.

Step 5: Least cost is 40. $x_{31} = \min(3, 5) = 3$. R_3 - completes

Step 6: Allocate 2 to cell (2, 1)

Total no of allocations = 3 + 4 - 1 = 6

∴ The solution is initial basic feasible.

Cost = (7 × 10) + (2 × 70) + (7 × 40) + (3 × 40) + (8 × 8) + (7 × 20)

= <u>814</u>

---

② Find an initial solution for the following TP by matrix-minima

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Demand	20	40	30	10	

Solution

Supply = 30 + 50 + 20 = 100

Demand = 20 + 40 + 30 + 10 = 100

→ TP is balanced

	D_1	D_2	D_3	D_4	Supply
O_1	•20 1	2	•10 1	4	~~30~~ ~~10~~ 0
O_2	3	•20 3	•20 2	•10 1	~~50~~ ~~40~~ ~~20~~ 0
O_3	4	•20 2	5	9	~~20~~ 0
Demand	~~20~~ 0	~~40~~ ~~20~~ 0	~~30~~ ~~20~~ 0	~~10~~ 0	

Step 1: Least cost is 1. We shall select (1, 1). +
x_{11} = min (30, 20) = 20. C_1 - completes.

Step 2: Least cost is 1. x_{13} = min (10, 30) = 10. R_1 - completes

Step 3: Least cost is 1. x_{24} = min (50, 10) = 10. C_4 - completes

Step 4: Least cost is 2. x_{23} = min (20, 40) = 20. C_3 - completes

Step 5: Least cost is 2. x_{32} = min (20, 40) = 20. R_3 - completes

Step 6: Allocate 20 to (2, 2).

Total no of allocations = 6 = 3 + 4 - 1

∴ The solution is initial basic feasible

Cost = (20×1) + (10×1) + (20×3) + (20×2) + (10×1) + (20×2)

= 180

---

③ Using matrix-minima method, find an initial solution of TP.

	D1	D2	D3	Supply
O1	5	7	8	70
O2	4	4	6	30
O3	6	7	7	50
Demand	65	42	43	

Solution:

Supply = 70 + 30 + 50 = 150

Demand = 65 + 42 + 43 = 150

⇒ TP is balanced

	D1	D2	D3	Supply
O1	•65 5	•5 7	8	70
O2	4	•30 4	6	30
O3	6	•7 7	•43 7	50
Demand	65	42	43	

Step 1: Least cost is 4. $x_{22} = \min(30, 42) = 30$. R_2-completes

Step 2: Least cost is 5. $x_{11} = \min(70, 65) = 65$. C_1-completes

Step 3: Least cost is 7. $x_{12} = \min(5, 12) = 5$. R_1-completes

Step 4: Least cost is 7. $x_{32} = \min(50, 7) = 7$. C_2-completes

Step 5: Allocate 43 to (3,3)

Total no. of allocations $= 5 = 3+3-1$

∴ The solution is initial basic feasible

$$\text{Cost} = (65\times5) + (5\times7) + (30\times4) + (7\times7) + (43\times7) = \underline{\underline{830}}$$

---

④ Solve the following TP by matrix-minima method.

	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	

Solution:

$$\text{Supply} = 5 + 8 + 7 \ast 14 = 34$$

$$\text{Demand} = 7 + 9 + 18 = 34$$

$\Rightarrow$ TP is balanced.

	D_1	D_2	D_3	Supply
O_1	2	7 •2	4 •3	~~5~~ ~~2~~ 0
O_2	3	3	1 •8	~~8~~ 0
O_3	5	4 •7	7	~~7~~ 0
O_4	1 •7	6	2 •7	~~14~~ ~~7~~ 0
Demand	~~7~~ 0	~~9~~ ~~2~~ 0	~~18~~ ~~10~~ ~~3~~ 0	

Step 1: Least cost is 1. $x_{41} = \min(14, 7) = 7$. C_1-completes

Step 2: Least cost is 1. $x_{23} = \min(8, 18) = 8$. R_2-completes

Step 3: Least cost is 2. $x_{43} = \min(10, 7) = 7$. R_4-completes

Step 4: Least cost is 4. $x_{32} = \min(7, 9) = 7$. R_3-completes

Step 6 : ~~Least~~ ~~cos~~ Allocate 2 to (1,2).

Total no. of allocations = 6 = 4 + 3 - 1

∴ The solution is initial basic feasible

Cost = $(2\times7) + (3\times4) + (8\times1) + (7\times4) + (7\times1) + (7\times2) = \underline{\underline{83}}$

OPTIMAL SOLUTION BY MODIFIED DISTRIBUTION METHOD

Working Rule:

Step 1 : Find an initial basic feasible solution (preferably using VAM technique). Assign variables $u_1, u_2, u_3 \ldots$ to rows, $v_1, v_2, v_3 \ldots$ to columns.

Step 2: Consider the row or the column with maximum number of allocations. If there is a tie, break it arbitrarily. Assign u or $v = 0$. Find the other values of u_i and v_j using the formula $\boxed{c_{ij} = u_i + v_j}$

NOTE: u_i and v_j values are calculated with the help of the cells where there are allocations.

Step 3: Find $u_i + v_j$ for the cells without allocations (These entries are written at the bottom right corner).

Step 4: Find $d_{ij} = c_{ij} - (u_i + v_j)$ for the cells where there are no allocations. (These d_{ij} values are written at the top right corner).

Step 5: We apply the following optimality test

(i) If all $d_{ij} \geq 0$, then the current solution is the optimal solution.

(ii) If atleast one d_{ij} is -ve, then the solution is not optimal. To find the optimal solution, go to step6.

Step 6: Select the cell with the least negative d_{ij}. (Tie is broken arbitrarily). Allocate an unknown quantity, say θ to this cell. Construct a loop that starts and ends with this cell. Draw the loop in such a way that the corners of the loop are the cells with allocations (A loop can have a maximum of 2 allocated cells in a row or column).

The amount of θ is added and subtracted from the corner cells of the loop in such a way that the demand and supply values remain satisfied.

Step 7: Assign the largest possible values to θ in such a way that the value of one basic cell (cell with allocation) becomes zero.

Step 8: Return to step 2 and repeat the procedure until optimal solution is obtained i.e. all d_{ij}'s are non-negative.

PART-B

UNIT-6

TRANSPORTATION & ASSIGNMENT PROBLEMS

I METHOD:

(1) NORTH-WEST CORNER METHOD:

Find the initial Basic feasible solution by NWcorner method for the following:

PROBLEM:

Warehouse / Factory	W1	W2	W3	W4	Supply
F1	19	30	50	19	07
F2	70	30	40	60	09
F3	40	8	70	20	18
demand	5	8	07	14	34 34

Solution:

Step 1: To check whether
SUPPLY = DEMAND

Step 2:

	W1	W2	W3	W4	Supply
F1	19 5	30 2	50	19	07/2/0
F2	70	30 6	40 3	60	09/3/0
F3	40	8	70 4	20 14	18/14/0
demand	5/0	8/6/0	07/4/0	14/0	34 34

∴ Basic feasible solution = Total cost = Z =

$$= (19\times5)+(30\times2)+(30\times6)+(40\times3)+(70\times4)+(20\times14)$$

$$\therefore \boxed{Z = 1015}$$

II METHOD: Row Minimum Method:

Find the BF solution for the above problem using Row minimum method

Solution:

	W1	W2	W3	W4	Supply
F1	19	30	60	10 7	7/0
F2	70	30 8	40 1	60	9/1/0
F3	40 5	8	70 6	20 7	18/11/6/0
demand	5/0	8/0	7/6/0	14/7/0	

BF Solution = Total cost = $Z = (19 \times 7) + (30 \times 8) + (40 \times 1) + (40 \times 5) + (70 \times 6) + (20 \times 7)$

$\therefore \boxed{Z = 1110}$

Procedure:

(1) Search for the minimum cost in the matrix row wise and allocate to them accordingly depending on the supply and demand mentioned.

(2) Supposingly, if either of supply or demand is zero, search for the next minimum cost in the matrix and continue the allocation process until all supplies and demands are zeros. (i.e., if the supply of the particular row becomes zero, it is as good as eliminating that row. Similarly if the demand of particular becomes zero after allocation, it is as good as eliminating that particular column).

III. METHOD: COLUMN MINIMUM METHOD:

Find the BF Solution for the following above problem using column minimum method.

Solution:

	W1	W2	W3	W4	Supply
F1	19 5	30	50	19 2	7/2/0
F2	70	30	40 7	60 2	9/2/0
F3	40	8 8	70	20 10	18/10/0
demand	5/0	8/0	7/0	14/12/10/0	

$\therefore$ BF solution = Total cost = $Z = (19\times5)+(8\times8)+(40\times7)+(19\times2)+(60\times2)+(20\times10)$

$\therefore \boxed{Z = 779}$

Procedure:

(1) Select the minimum cost, column wise and allocate to them accordingly depending on the supply & demand provided in the problem.

(2) If the supply/demand becomes zero after allocation, it is as good as elimenating that particular row/column respectively.

(3) Allocation is made until all supplies/demands becomes zeros.

IV. METHOD: MATRIX MINIMUM METHOD:

	W1	W2	W3	W4	Supply
F1	19	30	50	10 7	7/0
F2	70 2	30	40 7	60	9/2/0
F3	40 3	8 8	70	20 7	18/10/3/0
demand	5/2	8/0	7/0	14/7/0	

∴ BF solution = Total cost = $Z = (10\times7)+(70\times2)+(40\times7)+(40\times3)+(8\times8)+(20\times7)$

∴ $\boxed{Z = 814}$

Procedure:

(1) Search for the whole matrix with the least minimum cost and allocate it depending on the demand and supply given.

(2) As and when the supply/demand becomes zero after allocating, cancel that particular row/column respectively and thus no more further allocations should be made to those cancelled rows/columns.

(3) Continue the same above 2 steps until the demand/supply becomes zeros.

V METHOD: VOGEL'S APPROXIMATION METHOD:

For the above problem, find BF solution using Vogel's approximation method:

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I Iteration:

	W1	W2	W3	W4	Supply	Penalty
F1	19	30	50	10	7	9
F2	70	30	40	60	9	10
F3	40	8 8	70	20	18/10	12
demand	5	8/0	7	14	34 \ 34	
Penalty	21	22 ↑	10	10		

Z =(8×8)

II Iteration: After eliminating 'W2' column.

	W1	W3	W4	Supply	Penalty
F1	19 5	50	10	7/2	9
F2	70	40	60	9	20
F3	40	70	20	18/10	20
demand	5/0	7	14		
Penalty	21 ↑	10	10		

Z = (8×8) + (19×5)

III Iteration: After eliminating column 'W1'

	W3	W4	Supply	Penalty
F1	$\boxed{50}$	$\boxed{10}$	7/2	40
F2	$\boxed{40}$	$\boxed{60}$	9	20
F3	$\boxed{70}$	$\boxed{20}$ 10	18/10/0	㊿ ←
demand	7	14/4		
Penalty	10	10		

$Z = (8\times8) + (19\times5) + (20\times10)$

IV Iteration: After eliminating row 'F3'

	W3	W4	Supply	Penalty
F1	$\boxed{50}$	$\boxed{10}$ 2	7/2/0	40
F2	$\boxed{40}$	$\boxed{60}$	9	20
demand	7	14/4/2		
Penalty	10	㊿ ↑		

$Z = (8\times8) + (19\times5) + (20\times10) + (10\times2)$

V Iteration: After eliminating row 'F1'.

	W3	W4	Supply	Penalty
F2	$\boxed{40}$ 7	$\boxed{60}$ 2	9/2/0	-
demand	7/0	14/4/2/0		
Penalty	-	-		

∴ BF solution = Total cost = Z

∴ Z = (8×8) + (19×5) + (20×10) + (10×2) + (40×7) + (60×2)

∴ $\boxed{Z = 779}$

(2) PROBLEM-2: Find BF solution:

Destination

Source

	D1	D2	D3	D4	Supply
S1	3	7	6	4	5
S2	2	4	3	2	2
S3	4	3	8	5	3
demand	3	3	2	2	10 10

Step 1: To check Supply = Demand

Here Supply = Demand = 10

I METHOD: NORTH WEST METHOD:

	D1	D2	D3	D4	Supply
S1	$\boxed{3}$ 3	$\boxed{7}$ 2	$\boxed{6}$	$\boxed{4}$	5/2/0
S2	$\boxed{2}$	$\boxed{4}$ 1	$\boxed{3}$ 1	$\boxed{2}$	2/1/0
S3	$\boxed{4}$	$\boxed{3}$	$\boxed{8}$ 1	$\boxed{5}$ 2	3/2/0
demand	3/0	3/1/0	2/1/0	2/0	10 10

∴ BF solution = Z = (3×3) + (7×2) + (4×1) + (3×1) + (8×1) + (5×2)

∴ $\boxed{Z = 48}$

II METHOD: VOGEL'S APPROXIMATION METHOD:

I Iteration:

	D1	D2	D3	D4	Supply	Penalty
S1	3	7	6	4	5	1
S2	2	4	3 2	2	2/0	0
S3	4	3	8	5	3	1
demand	3	3	2/0	2	10 / 10	
Penalty	1	1	③ ↑	2		

$Z = (3 \times 2)$

II Iteration: After eliminating column 'D3'

	D1	D2	D4	Supply	Penalty
S1	3	7	4	5	1
S2	2	4	2 0	2/0	0
S3	4	3	5	3	1
demand	3	3	2		
Penalty	1	1	② ↑		

$Z = (3 \times 2) + (2 \times 0)$

S1
S3
demai
Penal

IV I

der

∴ BF sol

NOTE:

III Iteration: After eliminating row 'S2', we get

	D1	D2	D4	Supply	Penalty
S1	3	7	4	5	1
S3	4	3 3	5	3/0	1
demand	3	3/0	2		
Penalty	1	④ ↑	1		

$Z = (3\times2)+(2\times0)+(3\times3)$

IV Iteration: After eliminating row 'S3', we get

	D1	D2	D4	Supply
S1	3 3	7 0	4 2	5/2/0
demand	3	0	2	

∴ BF solution $= Z = (3\times2)+(2\times0)+(3\times3)+(3\times3)+(7\times0)+(4\times2)$

$$\therefore \boxed{Z = 32}$$

NOTE: No of Basic Variables = Non-degenerate $= m+n-1$
(or) (No of allocations made)

where $m \longrightarrow$ no of rows

$n \longrightarrow$ no of columns.

Assignment Problems:

(3) PROBLEM 3:

Solve by Vogel's approximation method to find the Basic feasible solution for the below given problems.

	D1	D2	D3	D4	D5	Supply
S1	2	11	10	3	7	4
S2	1	4	7	2	1	8
S3	3	9	4	8	12	9
Demand	3	3	4	5	6	

Step 1: Here Supply = Demand = 21

I Iteration:

	D1	D2	D3	D4	D5	Supply	Penalty
S1	2	11	10	3	7	4	1
S2	1	4	7	2	1 6	8/2	0
S3	3	9	4	8	12	9	1
demand	3	3	4	5	6/0	21 \ 21	
Penalty	1	5	3	1	⑥ 4		

Z = ([illegible])(1×6)

II It

S1
S2
S3
dema
Penal

III Ite

S1
S3
dema
Penal

II Iteration: After eliminating the column `D5', we get

	D1	D2	D3	D4	Supply	Penalty
S1	2	11	10	3	4	1
S2	1	4 2	7	2	8/2/0	1
S3	3	9	4	8	9	1
demand	3	3/1	4	5		
Penalty	1	⑤ ↑	3	1		

$Z = (1 \times 6) + (4 \times 2)$

III Iteration: After eliminating the ~~column~~ row `S2', we get

	D1	D2	D3	D4	Supply	Penalty
S1	2	11	10	3	4	1
S3	3	9	4 4	8	9/5	1
demand	3	3/1	4/0	5		
Penalty	1	2	⑥ ↑	5		

$Z = (1 \times 6) + (4 \times 2) + (4 * 4)$

IV. Iteration: After eliminating the column 'D3',

	D1	D2	D4	Supply	Penalty
S1	2	11	3	4	1
S3	3 3	9	8	9/5/2	⑤
demand	3/0	3/1	5		
Penalty	1	2	5		

$Z = (1\times6) + (4\times2) + (4\times4)\times(3\times3)$

V Iteration: After eliminating the column 'D1'

	D2	D4	Supply	Penalty
S1	11	3 4	4/0	⑧ ← 4
S3	9	8	9/5/2	1
demand	3/1	5/1		
Penalty	2	5		

$Z = (1\times6) + (4\times2) + (4\times4) + (3\times4)\times(3\times3)$

VI Iteration: After eliminating the row 'S1', we get

	D2	D4	Supply	Penalty
S3	9 1	8 1	9/5/2/0	–
demand	3/1/0	5/1/0		
Penalty	–	–		

∴ BF Solution: $Z = (1\times6) + (4\times2) + (4\times4) + (3\times4) + (9\times1) + (8\times1)\times(3\times3)$

∴ $\boxed{Z = 68}$

II METHOD: NORTH WEST CORNER METHOD:

	D1	D2	D3	D4	D5	Supply
S1	$\boxed{2}$ 3	$\boxed{11}$ 1	$\boxed{10}$	$\boxed{3}$	$\boxed{7}$	4/1/0
S2	$\boxed{1}$	$\boxed{4}$ 2	$\boxed{7}$ 4	$\boxed{2}$ 2	$\boxed{1}$	8/6/2/0
S3	$\boxed{3}$	$\boxed{9}$	$\boxed{4}$	$\boxed{8}$ 3	$\boxed{12}$ 6	9/6/0
demand	3/0	3/2/0	4/0	5/3/0	6/0	21 21

$\therefore$ BF solution $= Z = (2\times3)+(11\times1)+(4\times2)+(7\times4)+(2\times2)+(8\times3)+(12\times6)$

$\therefore \boxed{Z=153}$

ASSIGNMENT 2:

PROBLEM: Find the BF solution for the following problem:

	D1	D2	D3	Supply
S1	13	15	16	17
S2	7	11	2	12
S3	19	20	9	16
demand	14	8	23	

Solution:

Step 1: Supply = Demand = 45.

x (3x3)

I METHOD: VOGEL'S APPROXIMATION METHOD:

I Iteration:

	D1	D2	D3	Supply	Penalty
S1	13	15	16	17	3
S2	7	11	2	12	5
S3	19	20	9 16	16/0	⑩ ←
demand	14	8	23/7	45 / 45	
Penalty	6	4	7		

Z = (9×16)

II Iteration: After eliminating the row 'S3'

	D1	D2	D3	Supply	Penalty
S1	13	15	16	17	3
S2	7	11	2 7	12/5	5
demand	14	8	23/7/0		
Penalty	6	4	⑭ ↑		

Z = (9×16) + (2×7)

III Iteration: After eliminating the column 'D3'

	D1	D2	Supply	Penalty
S1	13	15	17	2
S2	7 5	11	12/5/0	4
demand	14/9	8		
Penalty	⑥ ↑	4		

Z = (9×16) + (2×7) + (7×5)

IV Iteration: After eliminating the row 'S2', we get

	D1	D2	Supply
S1	$\boxed{13}$ 9	$\boxed{15}$ 8	17/8/0
demand	14/9/0	8/0	

$\therefore$ Z = BF solution = (9X16) + (2X7) + ([illegible]X5) + (13X9) + (15X8)

$\therefore$ $\boxed{Z = 430}$

II METHOD : NORTH WEST CORNER METHOD:

	D1	D2	D3	Supply
S1	$\boxed{13}$ 14	$\boxed{15}$ 3	$\boxed{16}$	17/3/0
S2	$\boxed{7}$	$\boxed{11}$ 5	$\boxed{2}$ 7	12/7/0
S3	$\boxed{19}$	$\boxed{20}$	$\boxed{9}$ 16	16/0
demand	14/0	8/5/0	23/16/0	45 45

$\therefore$ BF solution = Z = (13X14) + (15X3) + (11X5) + (2X7) + (9X16)

$\therefore$ $\boxed{Z = 440}$

* RUSSEL'S APPROXIMATION METHOD:

PROBLEM 1:

	D1	D2	D3	D4	D5	Supply	
S1	16	16	13 ㊵	22	17 ⑩	50/40/0	× ③
S2	14 ㉚	14	13 ㉚	19	15	60/30/0	× ⑤
S3	19 ⓪	19 ⑳	20	23 ㉚	M	50/30	
S4	M	0	M	0	0 ㊿	50/0	× ①
Demand	30/0	20/0	70/30/0	30/0	60/10/0	110 110	
			④ ×		× ②		

NOTE: Here M refers to ∞ (infinity)

Solution:

Iterations	$\overline{U_1}$	$\overline{U_2}$	$\overline{U_3}$	$\overline{U_4}$	$\overline{V_1}$	$\overline{V_2}$	$\overline{V_3}$	$\overline{V_4}$	$\overline{V_5}$	Largest negative Δ_{ij}	Allocation
	(rowwise max)				(columnwise max)						
1.	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$X_{45} = 50$
2.	22	19	M	–	19	19	20	23	M	$\Delta_{15} = -5-M$	$X_{15} = 10$
3.	22	19	23	–	19	19	20	23	–	$\Delta_{13} = -29$	$X_{13} = 40$
4	–	19	23	–	19	19	20	23	–	$\Delta_{23} = -26$	$X_{23} = 30$
5	–	19	23	–	19	19	–	23	–	$\Delta_{21} = -24$	$X_{21} = 30$
6.	–	–	23	–							

Iteration 1:

$$\boxed{\Delta_{ij} = C_{ij} - \overline{U}_i - \overline{V}_j}$$

$$\Delta_{11} = C_{11} - \overline{u}_1 - \overline{V}_1 = 16 - 22 - M = -5-M$$

$$\Delta_{12} = C_{12} - \overline{u}_1 - \overline{V}_2 = 16 - 22 - 19 = -25$$

$$\Delta_{13} = C_{13} - \overline{u}_1 - \overline{V}_3 = 13 - 22 - M = -9-M$$

$$\Delta_{14} = C_{14} - \overline{u}_1 - \overline{V}_4 = 22 - 22 - 23 = -23$$

$$\Delta_{15} = C_{15} - \overline{u}_1 - \overline{V}_5 = 17 - 22 - M = -5-M$$

$$\Delta_{21} = C_{21} - \overline{U}_2 - \overline{V}_1 = 14 - 19 - M = -5-M$$

$$\Delta_{22} = C_{22} - \overline{U}_2 - \overline{V}_2 = 14 - 19 - 19 = -24$$

$$\Delta_{23} = C_{23} - \overline{U}_2 - \overline{V}_3 = 13 - 19 - M = -6-M$$

$$\Delta_{24} = C_{24} - \overline{U}_2 - \overline{V}_4 = 19 - 19 - 23 = -23$$

$$\Delta_{25} = C_{25} - \overline{U}_2 - \overline{V}_5 = 15 - 19 - M = -4-M$$

$$\Delta_{31} = C_{31} - \overline{U}_3 - \overline{V}_1 = 19 - M - M = 19-2M$$

$$\Delta_{32} = C_{32} - \overline{U}_3 - \overline{V}_2 = 19 - M - 19 = -M$$

$$\Delta_{33} = C_{33} - \overline{U}_3 - \overline{V}_3 = 20 - M - M = 20-2M$$

$$\Delta_{34} = C_{34} - \overline{U}_3 - \overline{V}_4 = 23 - M - 23 = -M$$

$$\Delta_{35} = C_{35} - \overline{U}_3 - \overline{V}_5 = M - M - M = M$$

$$\Delta_{41} = C_{41} - \overline{U}_4 - \overline{V}_1 = M - M - M = -M$$

$$\Delta_{42} = C_{42} - \overline{U}_4 - \overline{V}_2 = 0 - M - 19 = -M-19$$

$$\Delta_{43} = C_{43} - \overline{U}_4 - \overline{V}_3 = M - M - M = -M$$

$$\Delta_{44} = C_{44} - \overline{U}_4 - \overline{V}_4 = 0 - M - 23 = -23-M$$

$$\Delta_{45} = C_{45} - \overline{U}_4 - \overline{V}_5 = 0 - M - M = -2M \checkmark$$

Now eliminate the row S_4, after allocating supply=50 to Δ_{45}, since the supply now becomes zero.

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× ⑤

× ①

ation

=50

10

40

30

=30

Iteration 2:

$\Delta_{11} = 16 - 22 - 19 = -25$

$\Delta_{12} = 16 - 22 - 19 = -25$

$\Delta_{13} = 13 - 22 - 20 = -29$

$\Delta_{14} = 22 - 22 - 23 = -23$

$\Delta_{15} = 17 - 22 - M = -5 - M$ ✓

$\Delta_{21} = 14 - 19 - 19 = -24$

$\Delta_{22} = 14 - 19 - 19 = -24$

$\Delta_{23} = 13 - 19 - 20 = -26$

$\Delta_{24} = 19 - 19 - 23 = -23$

$\Delta_{25} = 15 - 19 - M = -4 - M$

$\Delta_{31} = 19 - M - 19 = -M$

$\Delta_{32} = 19 - M - 19 = -M$

$\Delta_{33} = 20 - M - 20 = -M$

$\Delta_{34} = 23 - M - 23 = -M$

$\Delta_{35} = M - M - M = -M$

Now eliminate the column 'D5', after allocating the demand = 10 to it as the demand becomes zero after allocation.

Iteration 3:

$\Delta_{11} = 16 - 22 - 19 = -25$

$\Delta_{12} = 16 - 22 - 19 = -25$

$\Delta_{13} = 13 - 22 - 20 = -29$ ✓

$\Delta_{14} = 22 - 22 - 23 = -23$

$\Delta_{21} = 14 - 19 - 19 = -24$

$\Delta_{22} = 14 - 19 - 19 = -24$

$\Delta_{23} = 13 - 19 - 20 = -26$

$\Delta_{24} = 19 - 19 - 23 = -23$

$\Delta_{31} = 19 - 23 - 19 = -23$

$\Delta_{32} = 19 - 23 - 19 = -23$

$\Delta_{33} = 20 - 23 - 20 = -23$

$\Delta_{34} = 23 - 23 - 23 = -23$

Now eliminate the row 'S1' after allocating supply=40 to Δ_{13} as the supply becomes zero after allocation

Iteration 4:

$\Delta_{21} = 14-19-19 = -24$

$\Delta_{22} = 14-19-19 = -24$

$\Delta_{23} = 13-19-20 = -26$ ✓

$\Delta_{24} = 19-19-23 = -23$

$\Delta_{31} = 19-23-19 = -23$

$\Delta_{32} = 19-23-19 = -23$

$\Delta_{33} = 20-23-20 = -23$

$\Delta_{34} = 23-23-23 = -23$

Now eliminate the column D3, after allocating demand=30 to Δ_{23} as the supply becomes Zero after allocation.

Iteration 5:

$\Delta_{21} = 14-19-19 = -24$ ✓

$\Delta_{22} = 14-19-19 = -24$

$\Delta_{24} = 19-19-23 = -23$

$\Delta_{31} = 19-23-19 = -23$

$\Delta_{32} = 9-23-19 = -23$

$\Delta_{34} = 23-23-23 = -23$

Now final eliminate the row 'S2' after allocating Supply = Demand = 30. Since here both Demand and supply becomes zero after allocation, either of row 'S2' or column 'D1' can be eliminated.

$\therefore$ BF Solution = $Z = (0\times50)+(17\times10)+(13\times40)+(13\times30)+(14\times30)+(19\times0)+(19\times20)+(23\times30)$

$\therefore \boxed{Z = 2570}$

(1) Find the initial Basic feasible solution (optimal Solution) for the problem by Vogel's approximation method.

	W1	W2	W3	W4	Capacity (supply)
F1	19	30	50	10	7
F2	70	30	40	60	9
F3	40	8	70	20	18
warehouse required (demand)	5	8	7	14	34 34

Step 1: Supply = Demand = 34.

Solution: VOGEL'S APPROXIMATION METHOD: Iteration 1:

	W1	W2	W3	W4	Supply	Penalty
F1	19	30	50	10	7	9
F2	70	30	40	60	9	10
F3	40	8 8	70	20	18/10	12
demand	5	8/0	7	14	34 34	
Penalty	21	㉒ ↑	10	10		

Z = (8×8)

Itera

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F
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Iteration 2: After eliminating the column 'W2'

	W1	W3	W4	Supply	Penalty
F1	19 5	50	10	7/2	9
F2	70	40	60	9	20
F3	40	70	20	18/10	20
demand	5/0	7	14		
Penalty	㉑ ↑	10	10		

Z = (8×8) + (19×5)

Iteration 3: After eliminating the column 'w1'

	W3	W4	Supply	Penalty
F1	50	10	7/2	40
F2	40	60	9	20
F3	70	20 10	18/10/0	㊿ ←
demand	7	14/4		
Penalty	10	10		

Z = (8×8)+(19×5)+(20×10)

Iteration 4: After eliminating the row 'F3', we get

	W3	W4	Supply	Penalty
F1	50	10 2	7/2/0	40
F2	40	60	9	20
demand	7	14/4/2		
Penalty	10	㊿ ↑		

$Z = (8\times8) + (19\times5) + (20\times10) + (10\times2)$

Iteration 5: After eliminating the row 'F1', we get.

	W3	W4	Supply
F2	40 7	60 2	9/2/0
demand	7/0	14/4/2/0	

$Z = (8\times8) + (19\times5) + (20\times10) + (10\times2) + (40\times7) + (60\times2)$

$\therefore \boxed{Z = 779}$

Step 2 * To check whether the value of Z is optimal solution or not:

Non-degenerate $= m+n-1$
$= 3+4-1$
$= 6$ allocation.

Hence 6 allocations are made. Hence no non-degeneracy problem.

* Secondly, the allocation made should not form a loop.

	W1	W2	W3	W4	Supply	
F1	19 5	30	50	10 2	7	$u_1 = 10$
F2	70	30	40 7	60 2	9	$u_2 = 9$
F3	40	8 8	70	20 10	18/10	$u_3 = 20$
demand	5	8/0	7	14		

$V_1 = 9$ $V_2 = -12$ $V_3 = -20$ $V_4 = 0$

Step 3: Consider: $C_{ij} = u_i + V_j$

$$u_1 + V_1 = 19$$
$$u_1 + V_4 = 10$$
$$u_2 + V_3 = 40$$
$$u_2 + V_4 = 60$$
$$u_3 + V_2 = 8$$
$$u_3 + V_4 = 20$$

Now make $V_4 = 0$ ($\because$ V_4 column has the max no of allocations made)

If $V_4 = 0$,

$$u_1 = 10$$
$$V_1 = 9$$
$$u_2 = 60$$
$$u_3 = 20$$
$$V_3 = -20$$
$$V_2 = -12$$

Step 4: To find d_{ij} for all the locations in the above matrix where the allocation is not made.

$$d_{ij} = C_{ij} - u_i - V_j$$
$$d_{12} = C_{12} - u_1 - V_2 = 30 - 10 - (-12) = 32$$
$$d_{13} = C_{13} - u_1 - V_3 = 50 - 10 + 20 = 60$$
$$d_{21} = C_{21} - u_2 - V_1 = 70 - 60 - 9 = 1$$
$$d_{22} = C_{22} - u_2 - V_2 = 30 - 60 - 12 = -18$$
$$d_{31} = C_{31} - u_3 - V_1 = 40 - 20 - 9 = 11$$

$d_{33} = 70-20+20 = 70$

Consider the largest negative number in d_{ij}'s as the ENTERING BASIC VARIABLE.

$\therefore \boxed{d_{22} = -18 = \text{Entering Basic Variable}}$

Step 5: To find leaving Basic Variable

	W1	W2	W3	W4
D1	19 5	30	50	10 2
D2	70	30 +θ ②	40 7	60 2 -θ ⓪
D3	40	8 8 ⑥ -θ	70	20 10 ⑫ +θ

→ Leaving Basic Variable

Since d_{22} = entering Basic Variable, consider it as +θ.

Form a loop with the allocation made and mark alternate +θ and -θ s.

To find +θ value, consider -θ values in the formed loop and consider the least value amongst them as the value for $\theta = d_{22}$.

$\therefore \boxed{\text{Minimum value} = \theta = 2}$

For all +θ's add 2 and for all -θ's subtract 2 in the loop formed.

Whichever ever value becomes zero, then that is called leaving Basic Variable.

$\therefore \boxed{d_{24} = \text{leaving Basic Variable}}$

Step 5: Again find out $u_1, u_2, \ldots u_n$ & $v_1, v_2, \ldots v_n$ values. Then find out d_{ij}'s values until all the values of d_{ij}'s are positive. If all d_{ij}'s are positive, then it can be concluded that the solution obtained is the OPTIMAL SOLUTION.

To find $u_1, u_2, u_3, V_1, V_2, V_3$ values,

$u_1 + V_1 = 19$

$u_1 + V_4 = 10$

$u_2 + V_2 = 30$ (This is the new equation obtained)

$u_2 + V_3 = 40$

$u_3 + V_2 = 8$

$u_3 + V_4 = 20$

Now, Put $V_4 = 0$

$u_1 = 10$

$V_1 = 9$

$u_3 = 20$

$V_2 = -12$

$u_2 = 42$

$V_3 = -2$

$d_{12} = 30 - 10 + 12 = 32$

$d_{13} = 50 - 10 + 2 = 42$

$d_{21} = 70 - 42 - 9 = 19$

$d_{24} = 60 - 42 - 0 = 18$

$d_{31} = 40 - 20 + 2 = 22$

$d_{33} = 70 - 20 + 2 = 52$

Since all d_{ij}'s are positive, the optimal solution can be obtained

$Z = (19 \times 5) + (10 \times 2) + (30 \times 2) + (40 \times 7) + (8 \times 6) + (20 \times 12)$

$\therefore \boxed{Z = 743}$ is the optimal solution.

② PROBLEM 2: Find the initial Basic feasible Solution for the following by North-west Corner Method:

	S1	S2	S3	Supply
D1	2	7	4	5
D2	3	3	1	8
D3	5	4	7	7
D4	1	6	2	14
Demand	7	9	18	

Solution:

Step 1: Supply = Demand = 34

Step 2: NORTH WEST CORNER METHOD

	S1	S2	S3	Supply
D1	$\boxed{2}$ 5	$\boxed{7}$	$\boxed{4}$	5/0
D2	$\boxed{3}$ 2	$\boxed{3}$ 6	$\boxed{1}$	8/6/0
D3	$\boxed{5}$	$\boxed{4}$ 3	$\boxed{7}$ 4	7/4/0
D4	$\boxed{1}$	$\boxed{6}$	$\boxed{2}$ 14	14/0
Demand	7/2/0	9/3/0	18/14/0	34 34

Z = (2×5) + (3×2) + (3×6) + (4×3) + (7×4) + (2×14)

∴ Basic feasible solution = $\boxed{Z = 102}$

Step 3

D1

D2

D3

D4

Step 3: To find optimal solution.

Non-degenerate = m+n-1
$= 4+3-1$
$= 6$ (satisfied)

Consider: $c_{ij} = u_i + v_j$

$u_1 + v_1 = 2$
$u_2 + v_1 = 3$
$u_2 + v_2 = 3$
$u_3 + v_2 = 4$
$u_3 + v_3 = 7$
$u_4 + v_3 = 2$.

∴ Basic variables are $x_{11}, x_{21}, x_{22}, x_{32}, x_{33}, x_{43}$
(m+n-1 basic variables)

Consider and assume $u_2 = 0$.

∴ $V_1 = 3$
$u_1 = -1$
$V_2 = 3$
$u_3 = 1$
$V_3 = 6$
$u_4 = -4$

	S_1	S_2	S_3	
D1	2 5	7	4	$u_1 = -1$
D2	3 2	3 $-\theta$ 6 ②	1 ④ $+\theta$	$u_2 = 0$
D3	5	4 $+\theta$ 3 ⑦	7 $-\theta$ 4 ⓪	$u_3 = 1$
D4	1	6	2 14	$u_4 = -4$
	$V_1 = 3$	$V_2 = 3$	$V_3 = 6$	

← Leaving Basic Variable

$d_{12} = 7 - (-1) - 3 = 5$
$d_{13} = 4 - (-1) - 6 = -1$
$d_{23} = 1 - 0 - 6 = -5$ = Entering Basic Variable.
$d_{31} = 5 - 1 - 3 = 1$
$d_{41} = 1 - (-4) - 3 = 2$

d_{23} = Entering Basic Variable.

Here minimum value = $\boxed{\theta = 4}$

∴ d_{33} = leaving Basic Variable.

$Z = (2\times5) + (3\times2) + (3\times2) + (1\times4) + (4\times7) + (4\times2)$

∴ $\boxed{Z = 82}$

II. Iteration: To find $u_1, u_2, u_3, u_4, v_1, v_2, v_3$ values.

$u_1 + v_1 = 2$

$u_2 + v_1 = 3$

$u_2 + v_2 = 3$

$u_2 + v_3 = 1$

$u_3 + v_2 = 4$

$u_4 + v_3 = 2$

Put $u_2 = 0$

$v_1 = 3$

$u_1 = -1$

$v_2 = 3$

$u_3 = 1$

$v_3 = 1$

$u_4 = 1$

	S1	S2	S3	
D1	2 5	7	4	$u_1 = -1$
D2	3 2 $-\theta$ ⓪	3 2	1 $+\theta$ 4 ⑥	$u_2 = 0$
D3	5	4 7	7	$u_3 = 1$
D4	1 $+\theta$ ②	6	2 $-\theta$ 4 ⑫	$u_4 = 1$
	$v_1 = 3$	$v_2 = 3$	$v_3 = 1$	

d_{12}

d_{13}

d_{31}

d_{33}

d_{41}

d_{42}

obt

$d_{12} = 2-(-1)-3 = 0.$

$d_{13} = 4-(-1)-1 = 4$

$d_{31} = 5-(1)-3 = 1$

$d_{33} = 7-1-1 = 5$

$d_{41} = 1-1-3 = -3$ ⇒ Entering Basic Variable

$d_{42} = 2-1-1 = 0$

∴ d_{41} = Entering basic variable $= \theta$

Minimum value $= \boxed{\theta = 2}$

∴ d_{21} = leaving basic variable.

$Z = (2\times5)+(3\times2)+(1\times6)+(4\times7)+(1\times2)+(2\times12)$

∴ $\boxed{Z=76}$

Now again, find $u_1, u_2, u_3, u_4, V_1, V_2, V_3$ values

$u_1+V_1 = 2$
$u_2+V_2 = 3$
$u_2+V_3 = 1$
$u_3+V_2 = 4$
$u_4+V_1 = 1$
$u_4+V_3 = 2$

Put $u_2 = 0$
$V_2 = 3$
$V_3 = 1$
$u_3 = 1$
$u_4 = 1$
$V_1 = 0$
$u_1 = 2$

$d_{12} = 7-(-1)-(3) = 5$

$d_{13} = 4-(-1)-(1) = 4$

$d_{21} = 3-(0)-(3) = 0$

$d_{31} = 5-(1)-(3) = 1$

$d_{33} = 7-(1)-(1) = 5$

$d_{42} = 6-(1)-(3) = 2$

Since all d_{ij}'s are positive, the solution obtained above is optimal

∴ $\boxed{Z=76}$

(3) PROBLEM 3: (ASSIGNMENT):

	D1	D2	D3	D4	D5	Supply
S1	73	40	9	79	20	8
S2	62	93	96	8	13	7
S3	96	65	80	50	65	9
S4	57	58	29	12	87	3
S5	56	23	87	18	12	5
demand	6	8	10	4	4	32 32

find using lowest cost entry method ⓞⓡ Vogel's approximation method & find optimal Solution.

Solution:

I Step1: Supply = Demand = 32.

	D1	D2	D3	D4	D5	Supply	Penalty
S1	73	40	9 8	79	20	8/0	11
S2	62	93	96	8	13	7	5
S3	96	65	80	50	65	9	15
S4	57	58	29	12	87	3	17
S5	56	23	87	18	12	5	6
demand	6	8	10/2	4	4	34 34	
Penalty	1	17	⑳ ↑	4	1		

Z = (9x8)

Itera

S2
S3
S4
S5
demand
Penalty

Itera

S2
S3
S4
S5
demand
Penalty

Iteration 2: Eliminate the row 'S1'

	D1	D2	D3	D4	D5	Supply	Penalty
S2	62	93	96	8	13	7	5
S3	96	65	80	50	65	9	15
S4	57	58	29 2	12	87	3/1	17
S5	56	23	87	18	12	5	6
demand	6	8	10/2/0	4	4		
Penalty	1	35	(57) ↑	4	1		

$Z = (9\times8)+(29\times2)$

Iteration 3: Eliminate the column 'D3'

	D1	D2	D4	D5	Supply	Penalty
S2	62	93	8	13	7	5
S3	96	65	50	65	9	15
S4	57	58	12 1	87	3/1/0	(45) ←
S5	56	23	18	12	5	6
demand	6	8	4/3.	4		
Penalty	1	35	4	1		

$Z = (9\times8)+(29\times2)+(12\times1)$

Iteration 4: Eliminate the row 'S4'

	D1	D2	D4	D5	Supply	Penalty
S2	62	93	8	13	7	5
S3	96	65	50	65	9	15
S5	56	23 5	18	12	5/0	6
Demand	6	8/3	4/3	4		
Penalty	6	(42) ↑	10	1		

$Z = (9\times8) + (29\times2) + (12\times1) + (23\times5)$

Iteration 5: Eliminate the row 'S5'.

	D1	D2	D4	D5	Supply	Penalty
S2	62	93	8	13 4	7/3	5
S3	96	65	50	65	9	15
Demand	6	8/3	4/3	4/0		
Penalty	34	28	42	52		

$Z = (9\times8) + (29\times2) + (12\times1) + (23\times5) + (13\times4)$

Itera

S2

S3

Demand

Penalt

Itera

S3.

Demand

Step2:

Penalty
5
15
6
y

Iteration 6: Eliminate the column 'D5'

	D1	D2	D4	Supply	Penalty
S2	62	93	8 3	7/3/0	54
S3	96	65	50	9	15
Demand	6	8/3	4/3/0		
Penalty	34	28	42		

Iteration 7: Eliminate the row 'S2'

	D1	D2	D4	Supply
S3.	96 6	65 3	50 0	9/3/0
Demand	6/0	8/3/0	4/3/0	

$Z=(9\times8)+(29\times2)+(12\times1)+(23\times5)+(13\times4)+(8\times3)+(96\times6)+(65\times3)+(50\times0)$

$\boxed{Z=1104}$

Step2: To find optimal solution

Non-degenerate $= m+n-1$
$= 5+5-1$
$= \underline{\underline{9}}$

Consider : $C_{ij}=u_i+v_j$

$u_1+v_3=9$

$u_2+v_4=8$

$u_2+v_5=13$

$u_3+v_1=96$

$u_3+v_2=65$

$u_3+v_4=50$

$u_4+v_3=29$

$u_5+v_4=12$

	D1	D2	D3	D4	D5	
S1	73	40	9 8	79	20	$u_1=-58$
S2	62	93	96	8 3	13 4	$u_2=-42$
S3	96 6	65 3	80	50 0	65	$u_3=0$
S4	57	58	29 2	12 1	87	$u_4=-38$
S5	56	23 5	87	18	12	$u_5=-42$
	$V_1=96$	$V_2=65$	$V_3=67$	$V_4=50$	$V_5=55$	

Put $u_3=0$

$V_1=96$

$V_2=65$

$V_4=50$

$u_2=-42$

$u_4=-38$

$u_5=-42$

$V_3=67$

$u_1=-58$

$V_5=55$

$d_{ij}=c_{ij}-u_i-v_j$

$d_{13}=9-(-58)-67=0$

$d_{24}=8-(-42)-50=0$

$d_{25}=13-(-42)-55=0$

$d_{31}=96-(0)-96=0$

$d_{32}=65-(0)-65=0$

$d_{34}=50-0-50=0$

$d_{43}=29-(-38)-67=0$

$d_{44}=12-(-38)-50=0$

$d_{52}=23-(-42)-65=0$ ∴ All d_{ij}'s are positive. ∴ $\boxed{Z=1104}$

Sol

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Step1:
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× Problems where no of Basic Variables < non-degenerate:

Solving using Δ:

1) PROBLEM 1:

	X	Y	Z	Supply
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
demand	50	80	80	

$u_1 = -58$

$l_2 = -42$

$l_3 = 0$

$l_4 = -38$

$l_5 = -42.$

Step I:

Vogel's approximation method (Solution):

	X	Y	Z	Supply	Penalty
A	8	7	3	60	4
B	3 ㊿	8	9	70/20	⑤ ←
C	11	3	5	80	2
demand	50/0	80	80		
Penalty	5	4	2		

$Z = (3 \times 50)$

	Y	Z	Supply	Penalty
A	7	3	60	4.
B	8	9	70/20	1
C	3 (80)	5	80/0	2
demand	80/0	80		
Penalty	④4	2		

$Z = (3 \times 50) + (3 \times 80)$

A	3 ⑥⓪	60
B	9 ②⓪	70/20
C	5 ⓪	80/0
demand	80/60/0 .	

$$\therefore Z = (3\times50)+(3\times80)+(3\times60)+(9\times20)+(5\times0)$$

$$\therefore \boxed{Z = 750}$$

Step2: To find:

Non-degenerate = m+n-1

= 3+3-1

$\therefore$ $\boxed{\text{Non-degenerate} = 5}$

$\therefore$ No of Basic variable = 4 i.e ($x_{13}, x_{21}, x_{23}, x_{32}$)

Since $\boxed{\text{No of Basic variable} < \text{Non-degenerate}}$

Consider the following:

Step3:

	X	Y	Z	
A	8	7	3 60	$u_1 = 3$
B	3 50	8	9 20	$u_2 = 9$
C	11	3 80	5 Δ	$u_3 = 5$
	$V_1 = -6$	$V_2 = -2$	$V_3 = 0$	

(1) choose Δ, such a way that it should not from a closed loop.

(2) choose Δ, such that the cost at that particular location should be the least.

By taking above points to consideration,

choose x_{33} as the Δ.

Put $V_3 = 0$.

$u_1 + V_3 = 3$

$u_2 + V_1 = 3$

$u_2 + V_3 = 9$

$u_3 + V_2 = 3$

$u_3 + V_3 = 5$

$u_1 = 3$

$u_2 = 9$

$V_1 = -6$

$u_3 = 5$

$V_2 = -2$

Consider:

$d_{ij} = c_{ij} - u_i - v_j$

$d_{11} = 8 - 3 - (-6) = 11$

$d_{12} = 7 - 3 - (-2) = 6$

$d_{22} = 8 - 9 - (-2) = 2$

$d_{31} = 11 - 5 - (-6) = 12$.

All are positive.

∴ optimal solution = $\boxed{Z = 750}$

PROBLEM 2:

(2)

	w1	w2	w3	w4	w5	Supply
F1	4	3	1	2	6	40
F2	5	2	3	4	5	30
F3	3	5	6	3	2	20
F4	2	4	4	5	3	10
demand	30	30	15	20	5	

Solution:

Step 1: North-west corner method:

	w1	w2	w3	w4	w5	Supply
F1	㉚ 4	⑩ 3	1	2	6	40/10/0
F2	5	⑳ 2	⑩ 3	4	5	30/10/0
F3	3	5	⑤ 6	3 ⑮	2	20/15/0
F4	2	4	4	⑤ 5	3 ⑤	10/5/0
demand	30/0	30/20/0	15/5/0	20/5	5/0	100 100

optimal solution

$$Z = (4\times30)+(10\times3)+(2\times20)+(10\times3)+(6\times5)+(15\times3)+(5\times5)+(3\times5)$$

$$\therefore \boxed{Z = 335}$$

Step 2:

Non-degenerate = m+n-1

= 5+4-1

= 8.

Basic Variable = Non-degenerate

Step 2:

	W1	W2	W3	W4	W5	
D1	4 $-\theta$ ㉕ 30	3 $+\theta$ ⑮ 10	1	2	6	$u_1 = 0$
D2	5	2 $-\theta$ 20 ⑮	3 $+\theta$ ⑮ 10	4	5	$u_2 = -1$
D3	3	5	6 $-\theta$ 5 ⓪	3 $+\theta$ ⑳ 15	2	$u_3 = 2$
D4	2 5 $+\theta$	4	4	5 ⓪ 5	3 $-\theta$ 5	$u_4 = 4$
	$v_1 = 4$	$v_2 = 3$	$v_3 = 4$	$v_4 = 1$	$v_5 = -1$	

Consider

$u_1 + v_1 = 4$

$u_1 + v_2 = 3$

$u_2 + v_2 = 2$

$u_2 + v_3 = 3$

$u_3 + v_3 = 6$

$u_3 + v_4 = 3$

$u_4 + v_4 = 5$

$u_4 + v_5 = 3$

Put $u_1 = 0$

$v_1 = 4$

$v_2 = 3$

$u_2 = -1$

$v_3 = 4$

$u_3 = 2$

$v_4 = 1$

$u_4 = 4$

$v_5 = -1.$

$d_{ij} = c_{ij} - u_i - v_j.$

$d_{13} = 1 - 0 - 4 = -3$

$d_{14} = 2 - 0 - 1 = 1$

$d_{15} = 6 - 0 - (-1) = 7$

$d_{21} = 5 - (-1) - (4) = 2$

$d_{24} = 4 - (-1) - 1 = 4$

$d_{25} = 5 - (-1) - (-1) = 7$

$d_{31} = 3 - 2 - (4) = -3$

$d_{32} = 5 - 2 - (3) = 0$

$d_{35} = 2 - 2 - (-1) = 1$

$d_{41} = 2 - 4 - (4) = -6$ ⟶ Entering Basic variable (d_{41})

$d_{42} = 4 - 4 - (3) = -3$

$d_{43} = 4 - 4 - (4) = -4$

To find value of θ

$\min = (\theta - 30, \theta - 20, \theta - 5, \theta - 5)$

Here consider d_{44} as only the leaving Basic Variable, rather than d_{34}. because cost of $d_{44} < d_{34}$.

<u>II iteration</u>:

$u_1+v_1=4$

$u_1+v_2=3$

$u_2+v_2=2$

$u_2+v_3=3$

$u_3+v_3=6$

$u_3+v_4=3$

$u_4+v_1=2$

$u_4+v_5=3$.

Put $u_4=0$

$v_1=2$

$v_5=3$

$u_1=2$

$v_2=1$

$u_2=1$

$v_3=2$

$u_3=4$

$v_4=-1$

* ASSIGNMENT PROBLEM (HUNGARIAN METHOD):

(1) PROBLEM ①:

	A	B	C
1	120	100	80
2	80	90	110
3	110	140	120

Step 1: Find the least element in each row and subtract it with all the elements of that particular row. Matrix has to be a square matrix in Assignment Problem.

	A	B	C
1	40	20	0
2	0	10	30
3	0	30	10

Step 2: Find the least element in each column and subtract it with all the elements of that particular column.

	A	B	C
1	40	10	$\boxed{0}$
2	$\cancel{0}$	$\boxed{0}$	30
3	$\boxed{0}$	20	10

Step 3: Check rowwise/columnwise if it has one zero or not. If row/column has one zero, mark the assignment and cancel the other zero's in column/row respectively as shown above.

Step 4: check $\boxed{n = N}$

where n = no of assignments
N = no of rows

Conclusion

∴ optimal solution $Z = 80+90+110$

∴ $\boxed{Z = 280}$

∴ Assignments made are:

1 → C
2 → B
3 → A

---

② PROBLEM 2:

	A	B	C	D
P	8	26	17	11
Q	13	28	4	26
R	38	19	18	15
S	19	26	24	10

Step 1: Row-wise:

	A	B	C	D
P	0	18	9	3
Q	9	24	0	22
R	23	4	3	0
S	9	16	14	0

Step 2: Columnwise:

	A	B	C	D
P	$\boxed{0}$	14	9	3
Q	9	20	$\boxed{0}$	22
R	23	$\boxed{0}$	3	$\cancel{0}$
S	9	12	14	$\boxed{0}$

Here $N = n$

Conclusion: ∴ optimal solution: $Z = 8+4+19+10$

∴ $\boxed{Z = 41}$

∴ Assignment are:

P → A
Q → C
R → B

3) PROBLEM 3:

Step

	A	B	C	D	E
P	160	130	175	190	200
Q	135	120	130	160	175
R	140	110	155	170	185
S	50	50	80	80	110
T	[illegible]	35	70	80	105

Step 1: Row-wise

	A	B	C	D	E
P	30	0	45	60	70
Q	15	0	10	40	55
R	30	0	45	60	75
S	0	0	30	30	60
T	20	0	35	45	70

Step 2: columnwise ③

	A	B	C	D	E	
P	30	$\boxed{0}$	35	30	15	④
~~Q~~	~~15~~	~~0~~	$\boxed{0}$	~~10~~	~~0~~	
R	30	0	35	30	20	①
S	$\boxed{0}$	0	20	0	5	
T	20	0	25	15	15	②

Since here $n \neq N$

Step 3: Choose the minimum number in the matrix where the line does not passes.

Here ∴ $\boxed{\min = 15}$

Step 4: At the point of

(1) intersection of two lines → add no with min value

(2) where the lines just passes → keep the no as it is.

(3) where no line passes → subtract the number with the min value.

	A	B	C	D	E
P	15	$\cancel{0}$	20	15	$\boxed{0}$
Q	15	15	$\boxed{0}$	10	$\cancel{0}$
R	15	$\boxed{0}$	20	15	5
S	$\boxed{0}$	15	20	$\cancel{0}$	5
T	5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

∴ Now $n = N$

∴ optimal solution = $200+130+110+50+80$

∴ $\boxed{Z = 570}$

∴ Assignments are

P → E
Q → C
R → B
S → A
T → D

(4) PROBLEM:

	A	B	C	D	E	F
P	9	22	58	11	19	27
Q	43	78	72	50	63	48
R	41	28	91	37	45	33
S	74	42	27	49	39	32
T	36	11	57	22	25	18
U	3	56	53	31	17	28

Step 1: Row-wise:

	A	B	C	D	E	F
P	0	13	49	2	10	18
Q	0	35	29	7	20	5
R	13	0	63	9	17	5
S	47	15	0	22	12	5
T	25	0	46	11	14	7
U	0	53	50	28	14	25

Step 2: Columnwise:

	A ②	B	C	D	E	F ③	
P	$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
Q	$\cancel{0}$	35	29	5	10	$\cancel{0}$	①
R	13	$\cancel{0}$	63	7	7	$\boxed{0}$	⑤
S	47	15	$\boxed{0}$	20	2	$\cancel{0}$	
T	25	$\boxed{0}$	46	9	4	2	
U	$\boxed{0}$	53	50	26	4	20	④

Here $n \neq N$

$\boxed{\text{Min}=2}$

Step 3

F
Q
R
S
T
U

Step 4

Step 3:

	A	B	C	D	E	F
P	2	15	51	$\boxed{0}$	$\cancel{0}$	15
Q	$\cancel{0}$	35	29	3	8	$\cancel{0}$
R	13	$\cancel{0}$	63	5	5	$\boxed{0}$
S	47	15	$\boxed{0}$	18	$\cancel{0}$	$\cancel{0}$
T	25	$\boxed{0}$	46	7	2	2
U	$\boxed{0}$	53	50	24	2	20

①

Since here $n \neq N$

$\therefore \boxed{\text{Min} = 2}$

Step 4:

	A	B	C	D	E	F
P	4	17	51	$\boxed{0}$	$\cancel{0}$	17
Q	$\cancel{0}$	35	27	1	6	$\boxed{0}$
R	13	$\boxed{0}$	61	3	3	$\cancel{0}$
S	49	17	$\boxed{0}$	18	$\cancel{0}$	2
T	25	$\cancel{0}$	44	5	$\boxed{0}$	2
U	$\boxed{0}$	53	48	22	$\cancel{0}$	20

Here $n = N$

$\therefore$ optimal solution $Z = 11 + 48 + 28 + 27 + 25 + 3$

$\therefore \boxed{Z = 142}$

Assignments are:

P ⟶ D
Q ⟶ F
R ⟶ B
S ⟶ C
T ⟶ E
U ⟶ A

(5) PROBLEM:

	R1	R2	R3	R4
C1	9	14	19	15
C2	7	17	20	19
C3	9	18	21	18
C4	10	12	18	19
C5	10	15	21	16

Solution:

Step 1: In the above problem, there are 5 contractors and 4 Roads.

∴ Hence dummy Roads for each contractor has to be added to make matrix a SQUARE MATRIX.

	R1	R2	R3	R4	R5
C1	9	14	19	15	0
C2	7	17	20	19	0
C3	9	18	21	18	0
C4	10	12	18	19	0
C5	10	15	21	16	0

Step 2: Columnwise.

	R1	R2	R3	R4	R5
C1	2	2	1	$\boxed{0}$	$\cancel{0}$
C2	$\boxed{0}$	5	2	4	$\cancel{0}$
C3	2	6	3	3	$\boxed{0}$
C4	3	$\boxed{0}$	$\cancel{0}$	4	$\cancel{0}$
C5	3	3	3	1	$\cancel{0}$

Here $n \neq N$

∴ $\boxed{\text{Min} = 1}$

Step 3:

$$\begin{array}{c|ccccc}
 & R1 & R2 & R3 & R4 & R5 \\
\hline
C1 & 2 & 2 & 1 & \boxed{0} & 1 \\
C2 & \boxed{0} & 5 & 2 & 4 & 1 \\
C3 & 1 & 5 & 2 & 2 & \boxed{0} \\
C4 & 3 & \boxed{0} & \cancel{0} & 4 & 1 \\
C5 & 2 & 2 & 2 & \cancel{0} & \cancel{0}
\end{array}$$

Here $n \neq N$

$\therefore \boxed{\text{Min} = 1}$

Step 4:

$$\begin{array}{c|ccccc}
 & R1 & R2 & R3 & R4 & R5 \\
\hline
C1 & 1 & 1 & \boxed{0} & \cancel{0} & 1 \\
C2 & \boxed{0} & 5 & 2 & 5 & 2 \\
C3 & \cancel{0} & 4 & 1 & 1 & \boxed{0} \\
C4 & 3 & \boxed{0} & \cancel{0} & 5 & 2 \\
C5 & 1 & 1 & 1 & \boxed{0} & \cancel{0}
\end{array}$$

$\therefore$ Here $n = N$

$\therefore$ Optimal Solution $= Z = 19 + 7 + 0 + 12 + 16$

$\therefore \boxed{Z = 54 \text{ lakhs}}$

$\therefore$ Assignments are =

$$\boxed{\begin{array}{l} C1 \rightarrow R3 \\ C2 \rightarrow R1 \\ C3 \rightarrow R5 \\ C4 \rightarrow R2 \\ C5 \rightarrow R4 \end{array}}$$

UNIT- 7

GAME THEORY AND DECISION ANALYSIS

NOTE:
- \+ → A is winner
- − → B is winner
- 0 → fair Game.

* DOMINANCE STRATEGY:

1) PROBLEM 1:

		Player B (i)	(ii)	(iii)
Player A	I	3	5	4
	II	5	6	3
	III	8	7	9
	IV	4	2	8

Solution. Since A is winner:

Player A → tries to win maximum

Player B → tries to win minimum.

Here because Player A is winner (∵ so no −ve values in matrix)

Step1: While checking ~~row~~ wise, retain the maximum row.

While checking column wise, retain the minimum column.

Since $8>3$, $7>5$, $9>4$

$8>5$, $7>6$, $9>3$

$8>4$, $7>2$, $9>8$

eliminate I, II and IV row.

		(i)	(ii)	(iii) Player A
Player B	III	8	⑦	9

Since $7<8$, $7<9$

∴ $\boxed{\text{Value of game} = 7}$

(2) PROBLEM 2:

$$\begin{array}{c|cccc} & & \text{Player A} & & \\ & a & b & c & d \\ \hline 1 & -5 & 3 & 1 & 20 \\ \text{Player B} \quad 2 & 5 & 5 & 4 & 6 \\ 3 & -4 & 2 & 0 & -5 \end{array}$$

Step 1: Since $5 > -4$, $5 > 2$, $4 > 0$, $6 > -5$, eliminate 3rd row

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & -5 & 3 & 1 & 20 \\ 2 & 5 & 5 & 4 & 6 \end{array}$$

Step 2: Since $20 > 1$, $6 > 4$, eleminate d^{th} column.

$$\begin{array}{c|ccc} & a & b & c \\ \hline 1 & -5 & 3 & 1 \\ 2 & 5 & 5 & 4 \end{array}$$

Step 3: Since $3 > -5$, $5 \geq 5$ eliminate b^{th} column

$$\begin{array}{c|cc} & a & c \\ \hline 1 & -5 & 1 \\ 2 & 5 & 4 \end{array}$$

Step 4: Since $5 > -5$, $4 > 1$, eliminate 1^{st} row.

$$\begin{array}{c|cc} & a & c \\ \hline 2 & 5 & \textcircled{4} \end{array}$$

$\therefore \boxed{\text{Value of game} = 4}$

* SADDLE POINT METHOD:

1) PROBLEM 1:

$$\begin{array}{cc|ccc} & & & \text{Player B} & \\ & & a & b & c \\ \hline & 1 & -3 & -2 & 6 \\ \text{Player A} & 2 & 2 & 0 & 2 \\ & 3 & 5 & -2 & -4 \end{array}$$

Step 1:

		Player B a	b	c	Row minimum
	1	3	-2	6	-2
Player A	2	2	$\boxed{0}$	2	$\boxed{0}$
	3	5	-2	-4	-4
column max		5	$\boxed{0}$	6	

$\underline{V} = \max\min = 0$

$\bar{V} = \min\max = 0$

Since $\underline{V} = \bar{V} = 0$

$\therefore$ $\boxed{\text{Value of game} = 0}$ (fair game)

---

2) PROBLEM 2:

$$\begin{array}{cc|ccc} & & & \text{Player A} & \\ & & a & b & c \\ \hline & 1 & 3 & -4 & 8 \\ \text{Player B} & 2 & -8 & 5 & -6 \\ & 3 & 6 & -7 & 6 \end{array}$$

		Player A a	b	c	Row min
	1	3	-4	8	-4
Player B	2	-8	5	-6	-8
	3	6	-7	6	-7
col max		6	5	8	

$\underline{V} = \max\min = -4$

$\bar{V} = \min\max = 6$

$\therefore$ No saddle point.

$\therefore$ Hence the value of game is not predictable saddle point method.

3) PROBLE

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4) PROB

find

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3) **PROBLEM 3**: find the range of p & q. which will renders the entry (2,2) a saddle point for the game.

$$\begin{array}{c|ccc} & \multicolumn{3}{c}{\text{Players B}} \\ \hline & 2 & 4 & 5 \\ \text{Player A} & 10 & 7 & 9 \\ & 4 & p & 6 \end{array}$$

Solution:

$$\begin{array}{c|ccc|c} & \multicolumn{3}{c|}{\text{Player B}} & \text{Row min} \\ \hline & 2 & 4 & 5 & 2 \\ \text{Player A} & 10 & 7 & 9 & 7 \\ & 4 & p & 6 & 4 \\ \hline \text{Col max} & 10 & 7 & 6 & \end{array}$$

$\underline{V} = \max \min = 7$

$\overline{V} = \min \max = 7$

$\underline{V} = \overline{V} = 7$ (saddle point)-Given in the problem.

$\therefore$ Solution: $\boxed{\begin{array}{l} q \geqslant 7 \\ p \leqslant 7 \end{array}}$

---

4) **PROBLEM 4**:

$$\begin{array}{c|ccc} & \multicolumn{3}{c}{\text{Player B}} \\ \hline & \mu & 6 & 2 \\ \text{player A} & -1 & \mu & -7 \\ & -2 & 4 & \mu \end{array}$$

where pay-off matrix is strictly determinable.

find μ.

Solution: Conditions for determinable:

$V_R = V_c =$ Value of game.

	Player A			Row min
Player B	μ	6	2	2
	-1	μ	-7	-7
	-2	4	μ	-2
col max	-1	6	2	

$V = \max\min = 2$

$\overline{V} = \min\max = -1$

Suppose if $\mu = 0$

	Player A			Row min
Player B	0μ	6	2	0
	-1	0μ	-7	-7
	-2	4	0μ	-2
col max	0	6	2	

$V = \max\min = 0$

$\overline{V} = \min\max = 0$

Since $V = \overline{V}$ when $2 \geq \mu \geq -1$

$\therefore$ Solution : $\boxed{-1 \leq \mu \leq 2}$

---

③ PROBLEM-5:

	Player A		
Player B	-2	15	-2
	-5	-6	-4
	-5	20	-8

Solution:

	Player A			Row min
Player B	-2	15	-2	-2
	-5	-6	-4	-6
	-5	20	-8	-8
Col max	-2	20	-2	

$V = \max\min = -2$

$\overline{V} = \min\max = -2$

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1) PROBLEM

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$\therefore$ $\boxed{\text{Value of Game} = -2}$

$\therefore$ Player B is winner.

Player A $\longrightarrow$ I strategy

Player B $\longrightarrow$ I or III strategy

$\therefore$ Value of game for
Player A = -2
Player B = +2. (winner)

* **ALGEBRAIC METHOD (MIXED STRATEGY — for only 2×2 matrix)**

1) PROBLEM 1:

	Player A	
Player B	8	-3
	-3	1

Solution: Saddle point

	Player A (q)	(1-q)	Row min
Player B p	8	-3	-3
(1-p)	-3	1	-3
Col max	8	1	

$\underline{V} = \max \min = -3$

$\overline{V} = \min \max = 1$

Since $\underline{V} \neq \overline{V}$.

Algebraic method for Player B:

Let the probability that the player B chooses action I = p

Then let the probability that player B chooses action II = (1-p)

Expected pay-off gain of player B if A chooses strategy-I = 8p + (1-p)(-3)

Expected pay off gain of player B if A chooses strategy-II = -3p + 1(1-p)

$$8p+(1-p)(-3) = -3p+(1-p)$$
$$8p-3+3p = -3p+1-p$$
$$11p-3 = -4p+1$$
$$15p = 4$$
$$\boxed{p = 4/15}$$
$$\boxed{(1-p) = 11/15}$$

Out of 15 games, player A chooses 4 times of Strategy I and 11 times of strategy II.

Value of game $= 8p + (1-p)(-3)$

$$= 8\times\frac{4}{15} + (-3)\times\left(\frac{11}{15}\right)$$

$$= \frac{32}{15} - \frac{33}{15}$$

$$\therefore \boxed{\text{Value of game} = -1/15}$$

For Player A:

Let the prob that player A chooses action I $= q$.

Let the prob that player A chooses action-II $= (1-q)$

Expected pay-off gain of player A if B chooses strategy-I $= 8q + (1-q)(-3)$

Expected pay off gain of player A if B chooses strategy-II $= -3q + (1-q)(1)$

$$8q+(1-q)(-3) = -3q+(1-q)$$
$$8q-3+3q = -4q+1$$
$$15q = 4$$
$$\boxed{q = \frac{4}{15}}$$
$$\boxed{(1-q) = \frac{11}{15}}$$

$\therefore$ Value of game $= -3q + (1-q)$

$= -3 \times \frac{4}{15} + \frac{11}{15}$

$$\therefore \boxed{\text{Value of game} = -\frac{1}{15}}$$

Conclusion:

$\therefore (x_1, x_2) = (4/15, 11/15)$

$(y_1, y_2) = (4/15, 11/15)$

---

(2) PROBLEM 2:

$$\begin{array}{c|ccccc} & & & \text{Player A} & & \\ \hline & 4 & 4 & 2 & -4 & -6 \\ \text{Player B} & 8 & 6 & 8 & -4 & 0 \\ & 10 & 2 & 4 & 10 & 12 \end{array}$$

Step 1: Saddle point method:

$$\begin{array}{c|ccccc|c} & & & \text{Player A} & & & \text{Row min} \\ \hline & 4 & 4 & 2 & -4 & -6 & \cancel{-6} \\ \text{Player B} & 8 & 6 & 8 & -4 & 0 & -4 \\ & 10 & 2 & 4 & 10 & 12 & 2 \\ \hline \text{col. max} & 10 & 6 & 8 & 10 & 12 & \end{array}$$

$V = \max \min = 2$

$\bar{V} = \min \max = 6$

$\therefore V \neq \bar{V}$.

Step 2: Dominance Strategy.

Since $10 > 4, 12 > 4, 4 > 2, 10 > -4, 12 > -6$,

eliminate I row.

$$\begin{array}{c|ccccc} & & & \text{Player A} & & \\ \hline \text{Player B} & 8 & 6 & 8 & -4 & 0 \\ & 10 & 2 & 4 & 10 & 12 \end{array}$$

$0 > -4, 12 > 10$, eliminate 5th column.

$8 > -4, 10 \geq 10$, eliminate 1st column,

$8 > 6, 4 > 2$, eliminate 3rd column, so we

		Player A	
Player B		6	-4
		2	10

Step 3: Algebraic method:

		Player A q	(1-q)	Row min
Player B	(p)	6	-4	-4
	(1-p)	2	10	2
col max		6	10	

$\overline{V} = \max \min = 2$

$V = \min \max = 6$

$V \neq \overline{V}$. (∴ Not possible)

For Player B:

$$6p + (1-p)2 = -4p + (1-p)10$$

$$6p + 2 - 2p = -4p + 10 - 10p$$

$$4p + 2 = -14p + 10$$

$$18p = 8$$

$$\boxed{p = 4/9}$$

$$\boxed{(1-p) = 5/9}$$

$$\text{Value of game} = 6p + (1-p)2$$

$$= 6 \times \frac{4}{9} + 5/9 \times 2$$

$$= \frac{24}{9} + \frac{10}{9}$$

$$\therefore \boxed{\text{Value of game} = \frac{34}{9}}$$

For player A:

$$6q + (-4)(1-q) = 2q + 10(1-q)$$

$$6q - 4 + 4q = 2q + 10 - 10q$$

$$10q - 4 = -8q + 10$$

$$18q = 14$$

$$\boxed{q = 7/9}$$

$$\boxed{(1-q) = 2/9}$$

$\therefore$ Value of game $= 6q + (-4)(1-q)$

$$= 6 \times \frac{7}{9} + (-4)(2/9)$$

$$= \frac{42}{9} - \frac{8}{9}$$

$$\therefore \boxed{\text{Value of game} = \frac{34}{9}}$$

Conclusion:

$$\therefore (x_1, x_2, x_3) = (0, 4/9, 5/9)$$

$$(y_1, y_2, y_3, y_4, y_5) = (0, 7/9, 0, 2/9, 0)$$

---

3) PROBLEM 3:

$$\begin{array}{c|cccc|}
 & & \text{Player B} & & \\
\hline
 & 3 & 2 & 4 & 0 \\
 & 3 & 4 & 2 & 4 \\
\text{Player A} & 4 & 2 & 4 & 0 \\
 & 0 & 4 & 0 & 8 \\
\hline
\end{array}$$

Step 1: Saddle point method:

$$\begin{array}{c|cccc|c}
 & & \text{Player B} & & & \text{Row min} \\
\hline
 & 3 & 2 & 4 & 0 & 0 \\
 & 3 & 4 & 2 & 4 & 2 \\
\text{Player A} & 4 & 2 & 4 & 0 & 0 \\
 & 0 & 4 & 0 & 8 & 0 \\
\hline
\text{col max} & 4 & 4 & 4 & 8 &
\end{array}$$

$\bar{V} = \text{max min} = 2$

$\bar{V} \neq V$

$\therefore$ Not possible by saddle point.

Step 2: Dominance method;

Since $4 > 3$, $2 \geq 2$, $4 \geq 4$, $0 \geq 0$, eliminate I row.

$$\begin{array}{c|cccc} & & \text{Player B} & & \\ \hline & 3 & 4 & 2 & 4 \\ \text{Player A} & 4 & 2 & 4 & 0 \\ & 0 & 4 & 0 & 8 \end{array}$$

Since $3 > 2$, $4 \geq 4$, $0 \geq 0$

eliminate 1st column.

$$\begin{array}{c|ccc} & & \text{Player B} & \\ \hline & 4 & 2 & 4 \\ \text{Player A} & 2 & 4 & 0 \\ & 4 & 0 & 8 \end{array}$$

Eliminate 'min' in row, 'max' in column.

Since $4 > \frac{2+4}{2}$, $2 \geq \frac{4+0}{2}$, $4 \geq \frac{0+8}{2}$

eliminate II column.

$$\begin{array}{c|cc} & \text{Player B} & \\ \hline & 2 & 4 \\ \text{Player A} & 4 & 0 \\ & 0 & 8 \end{array}$$

Since $2 \geq \frac{4+0}{2}$, $4 \geq \frac{0+8}{2}$, eliminate II row

$$\begin{array}{c|cc} & \text{Player B} & \\ \hline & 4 & 0 \\ \text{Player A} & 0 & 8 \end{array}$$

		Player B		Row min
		q	(1-q)	
Player A	p	4	0	0
	(1-p)	0	8	0
col max		4	8	

$\overline{V} = \max\min = 0$

$V = \min\max = 4$

since $V \neq \overline{V}$

row,

For player A:

$4p + (1-p)(0)$ ——— ①

$0p + (1-p)(8)$ ——— ②

Equating ① & ②:

$4p + (1-p)(0) = 0p + (1-p)(8)$

$4p = 8 - 8p$

$12p = 8$

$\boxed{p = \frac{2}{3}}$

$\boxed{1-p = \frac{1}{3}}$

$\therefore$ Value of game $= 4p + (1-p)(0)$

$= 4(2/3) + (1/3)(0)$

$\therefore \boxed{\text{Value of game} = 8/3}$

For player B:

$4q + (1-q)(0)$ ——— ①

$0q + (1-q)(8)$ ——— ②

Equating ① and ②:

row

$4q + (1-q)0 = 0q + (1-q)8$

$4q = 8 - 8q$

$12q = 8$

$\boxed{q = \frac{2}{3}}$ $\boxed{1-q = 1/3}$

$\therefore$ Value of game $= 4q + (1-q)(0)$

$= 4 \times \frac{2}{3}$

$\therefore \boxed{\text{Value of game} = 8/3}$

∴ Conclusion =

$$(x_1, x_2, x_3, x_4) = (0, 0, 2/3, 1/3)$$

$$(y_1, y_2, y_3, y_4) = (0, 0, 2/3, 1/3)$$

(4) PROBLEM 4:

$$\begin{array}{c|ccc} & & \text{Player B} & \\ \hline & 5 & 0 & 2 \\ \text{Player A} & -1 & 8 & 6 \\ & 1 & 2 & 3 \end{array}$$

Step 1: Saddle point method:

$$\begin{array}{c|ccc|c} & & \text{Player B} & & \text{Row min} \\ \hline & 5 & 0 & 2 & 0 \\ \text{Player A} & -1 & 8 & 6 & -1 \\ & 1 & 2 & 3 & 1 \\ \hline \text{col max} & -1 & 0 & 2 & \end{array}$$

$\underline{V} = \max\min = 1$

$\bar{V} = \min\max = -1$

$V \neq \bar{V}$ ∴ Saddle point not found.

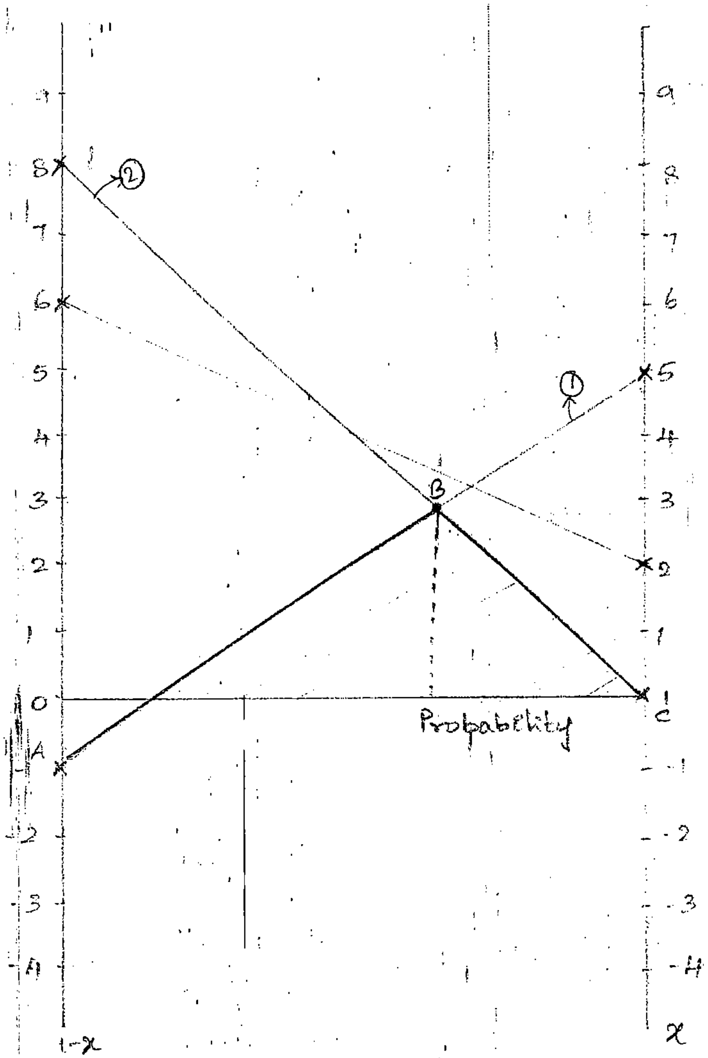
Step 2: Dominance method:

Since $\frac{5-1}{2} > 1$, $\frac{0+8}{2} \geq 2$, $\frac{2+6}{2} > 3$

Eliminate the III row.

$$\begin{array}{c|ccc} & & \text{Player B} & \\ \hline & 5 & 0 & 2 \\ \text{Player A} & -1 & 8 & 6 \end{array}$$

Apply graphical method to solve.



choose the maximum point in the lowest boundary for 2xm matrix. choose the point B which is intersection of lines ① and ②.

Player B

$$\begin{array}{c|cc} & y & 1-y \\ \hline x & 5 & 0 \\ (1-x) & -1 & 8 \end{array}$$

Player A

For player A:

$$5x + (-1)(1-x) = 0x + 8(1-x)$$

$$5x - 1 + x = 8 - 8x$$

$$6x - 1 = 8 - 8x$$

$$14x = 9$$

$$\boxed{x = 9/14}$$

$$\boxed{1-x = 5/14}$$

$$\text{Value of game} = 5x - (1-x)$$
$$= 5(9/14) - 5/14$$
$$= \frac{45}{14} - \frac{5}{14}$$
$$= \frac{\cancel{40}\ 20}{\cancel{14}\ 7}$$

$$\therefore \boxed{\text{Value of game} = 20/7}$$

For <u>player B</u>:

$$5y + 0(1-y) = -1y + 8(1-y)$$
$$5y = -1y + 8 - 8y$$
$$13y + 1y = 8$$
$$y = 8/14$$
$$\boxed{y = 4/7}$$
$$\boxed{(1-y) = 3/7}$$

$$\therefore \text{Value of game} = 5y + 0(1-y)$$
$$= 5(4/7) + 0$$

$$\therefore \boxed{\text{Value of game} = \frac{20}{7}}$$

∴ conclusion:

$$(x_1, x_2, x_3) = (9/14, 5/14, 0)$$
$$(y_1, y_2, y_3) = (4/7, 3/7, 0)$$

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* GRAPHICAL METHOD (for reduced matrix):

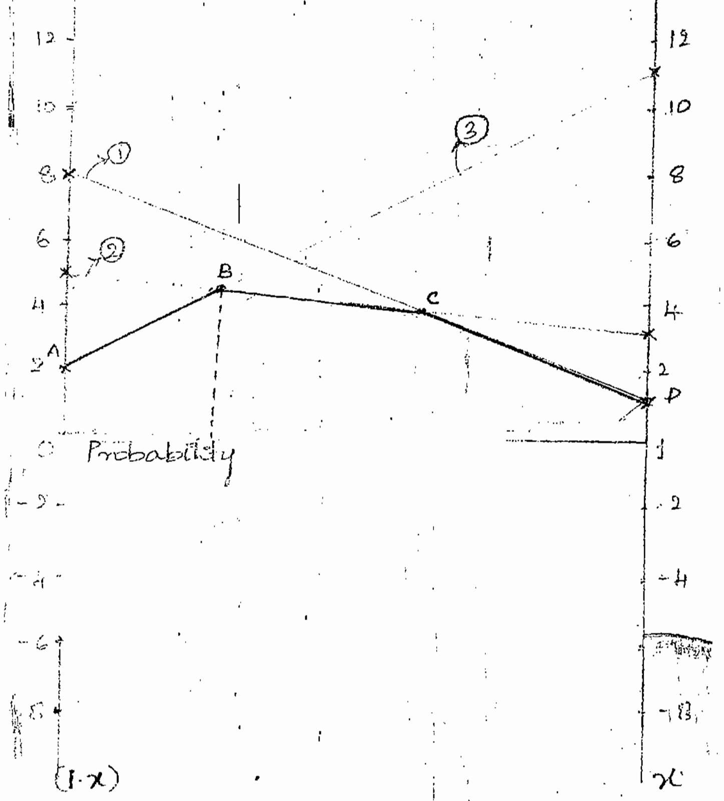
NOTE: (1) Graphical method is applicable for (2xm) or (nx2) matrices and is further Reduced to (2x2) matrix.

(2) for 2xm matrix, maximum value in the lower most boundary should be choosen.

(3) for nx2 matrix, minimum value in the uppermost boundary should be choosen.

1) PROBLEM 1:

		Player A		
Player B	x	1	3	11
	(1-x)	8	5	2



for (2xm) matrix, maximum value of the lower most boundary is choosen. Thus point B is choosen.

Point B is intersection of lines ② and ③.

Point B is formed by 2 lines $3x+5(1-x)$ and $11x+(1-x)2$

$$\therefore 3x+5(1-x) = 11x+(1-x)2$$

$$3x+5-5x = 11x+2-2x$$

$$5-2x = 9x+2$$

$$11x = 3$$

$$\boxed{x = 3/11}$$

$$\boxed{(1-x) = 8/11}$$

$$\begin{array}{c|c|cc} & & \text{Player A} & \\ & & y & 1-y \\ \hline \text{Player B} & x & 3 & 11 \\ & (1-x) & 5 & 2 \end{array}$$

Step 2: Algebraic method:

For player B: $3x+5(1-x) = 11x+2(1-x)$

$$3x+5-5x = 11x+2-2x$$

$$-2x+5 = 9x+2$$

$$11x = 3$$

$$\boxed{x = 3/11}$$

$$\boxed{(1-x) = 8/11}$$

$\therefore$ Value of game $= 3x+5(1-x)$

$$= \frac{3\times3}{11}+5(8/11)$$

$$= \frac{9}{11}+\frac{40}{11}$$

$$\therefore \boxed{\text{Value of game} = \frac{49}{11}}$$

For player A:

$$3y + 11(1-y) = 5y + 2(1-y)$$

$$3y + 11 - 11y = 5y + 2 - 2y$$

$$-8y + 11 = 3y + 2$$

$$11y = 9$$

$$\boxed{y = 9/11}$$

$$\boxed{1-y = 2/11}$$

∴ Value of game $= 3y + 11(1-y)$

$$= 3(9/11) + 11(2/11)$$

$$= \frac{27}{11} + \frac{22}{11}$$

∴ $\boxed{\text{Value of game} = \frac{49}{11}}$

∴ conclusion:

$$(x_1, x_2) = (3/11, 8/11)$$

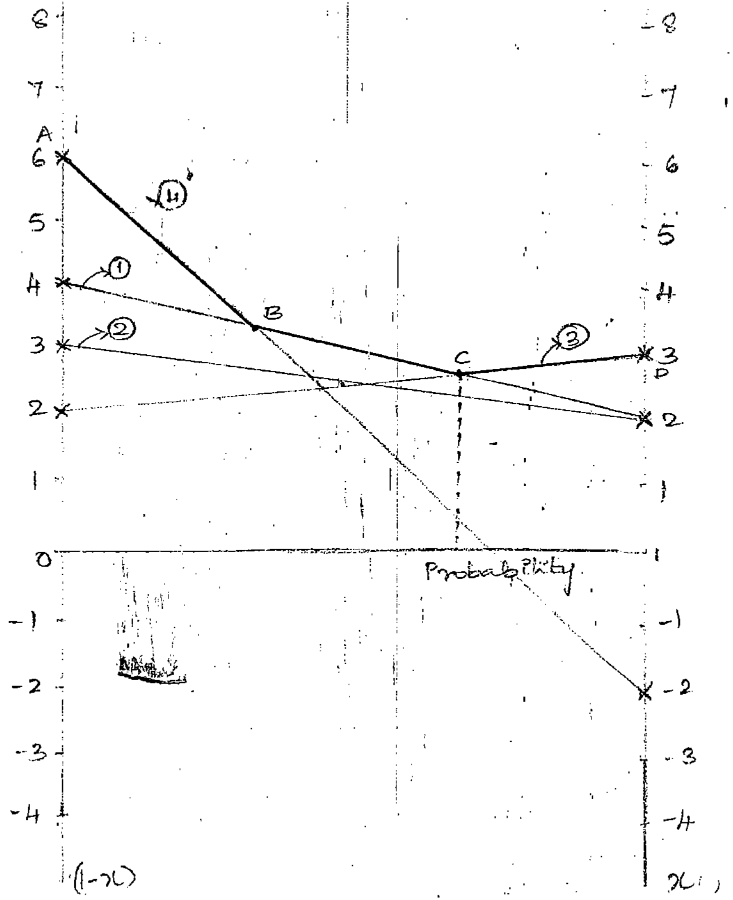
$$(y_1, y_2, y_3) = (0, 9/11, 2/11)$$

---

2) PROBLEM 2:

$$\begin{array}{c|cc} & \multicolumn{2}{c}{\text{Player B}} \\ \hline & 2 & 4 \\ \text{Player A} & 2 & 3 \\ & 3 & 2 \\ & -2 & 6 \end{array}$$

Step1: Graphical method for mx2 matrix:



for mx2 matrix, take least value from uppermost boundary.

Point c is choosen which is the intersection of lines ① and ③

		Player B	
		y	(1-y)
	x	2	4
Player A	(1-x)	3	2

for player A:

$$2x + 3(1-x) = 4x + 2(1-x)$$

$$2x + 3 - 3x = 4x + 2 - 2x$$

$$-x + 3 = 2x + 2$$

$$3x = 1$$

$$\boxed{x = 1/3}$$

$$\boxed{1-x = 2/3}$$

$\therefore$ Value of game $= 2x + 3(1-x)$

$= 2(1/3) + 3(2/3)$

$= \frac{2}{3} + \frac{6}{3}$

$\therefore \boxed{\text{Value of game} = \frac{8}{3}}$

For player B:

$2y + 4(1-y) = 3y + 2(1-y)$

$2y + 4 - 4y = 3y + 2 - 2y$

$-2y + 4 = y + 2$

$3y = 2$

$\boxed{y = 2/3}$

$\boxed{(1-y) = 1/3}$

$\therefore$ Value of game $= 2y + 4(1-y)$

$= 2(2/3) + 4(1/3)$

$= \frac{4}{3} + \frac{4}{3}$

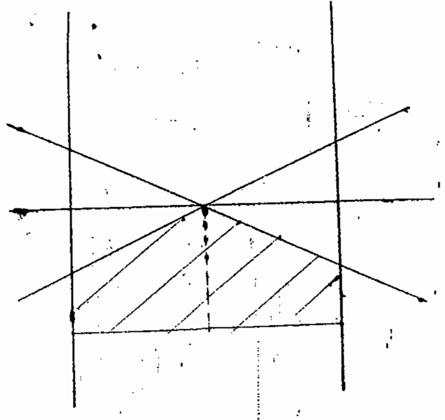
$\therefore \boxed{\text{Value of game} = 8/3}$

$\therefore$ Conclusion: $(x_1, x_2, x_3, x_4) = (1/3, 0, 2/3, 0)$

$(y_1, y_2) = (2/3, 1/3)$

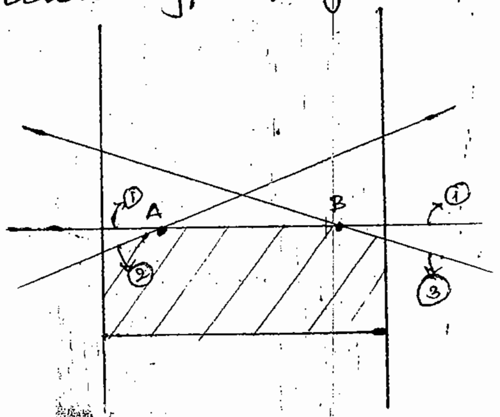
NOTE:

(1) For the below type of graph:



In the above case, choose any two lines at the point of intersection arbitrarily and the Value of game always remains same, even though any two points are chosen.

(2) For the below type of graph:



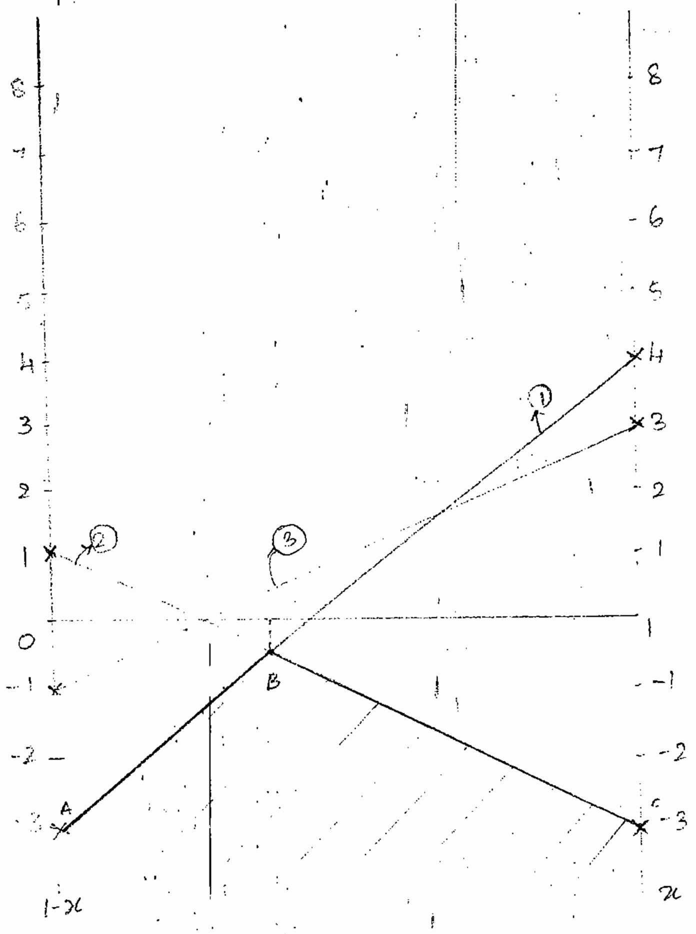
Here there are two maximum points A and B. Either of them can be choosen for further calculations.

3) PROBLEM 8:

		Player B	
Player A	4	-3	3
	3	1	-1

Procedure; • Saddle point method }
• Dominance method } all have to be checked in order.
• Algebraic method. }
• If solve by graphical method is given, else no need to check.

By Graphical method:



		Player B	
		y	1-y
Player A	x	4	-3
	1-x	3	1

For player A:

$$4x - 3(1-x) = -3x + 1(1-x)$$

$$4x - 3 + 3x = -3x + 1 - x$$

$$7x - 3 = -4x + 1$$

$$11x = 4$$

$$\boxed{x = \frac{4}{11}}$$

$$\boxed{(1-x) = 7/11}$$

$\therefore$ Value of game $= 4x - 3(1-x)$

$$= 4\left(\frac{4}{11}\right) - 3\left(7/11\right)$$

$$= \frac{16}{11} - \frac{21}{11}$$

$\therefore$ $\boxed{\text{Value of game} = \frac{-5}{11}}$

For player B:

$$4y - 3(1-y) = -3y + (1-y)$$

$$4y - 3 + 3y = -3y + 1 - y$$

$$7y - 3 = -4y + 1$$

$$11y = 4$$

$$\boxed{y = \frac{4}{11}}$$

$$\boxed{1-y = 7/11}$$

$\therefore$ Value of game $= 4y - 3(1-y)$

$$= 4(4/11) - 3(7/11)$$

$$= \frac{16}{11} - \frac{21}{11}$$

$\therefore$ $\boxed{\text{Value of game} = \frac{-5}{11}}$

$\therefore$ Conclusion :

$(x_1, x_2) = (4/11, 7/11)$

$(y_1, y_2, y_3) = (4/11, 7/11, 0)$

(4) PROBLEM 4:

Player B

		a	b	c	d
	1	8	15	-4	-2
Player A:-	2	19	15	17	16
	3	0	20	15	5

Solution: Saddle point method:

Player B

		a	b	c	d	Row min
	1	8	15	-4	-2	-2
Player A	2	19	15	17	16	15
	3	0	20	15	5	0
col max		19	20	17	16	

$\bar{V} = \max \min = 15$

$V = \min \max = 16$

Since $V \neq \bar{V}$, No saddlepoint

Step 2: Dominance method:

19 > 8, 15 ≥ 15, 17 > -4, 16 > -2, so eliminate 1st row.

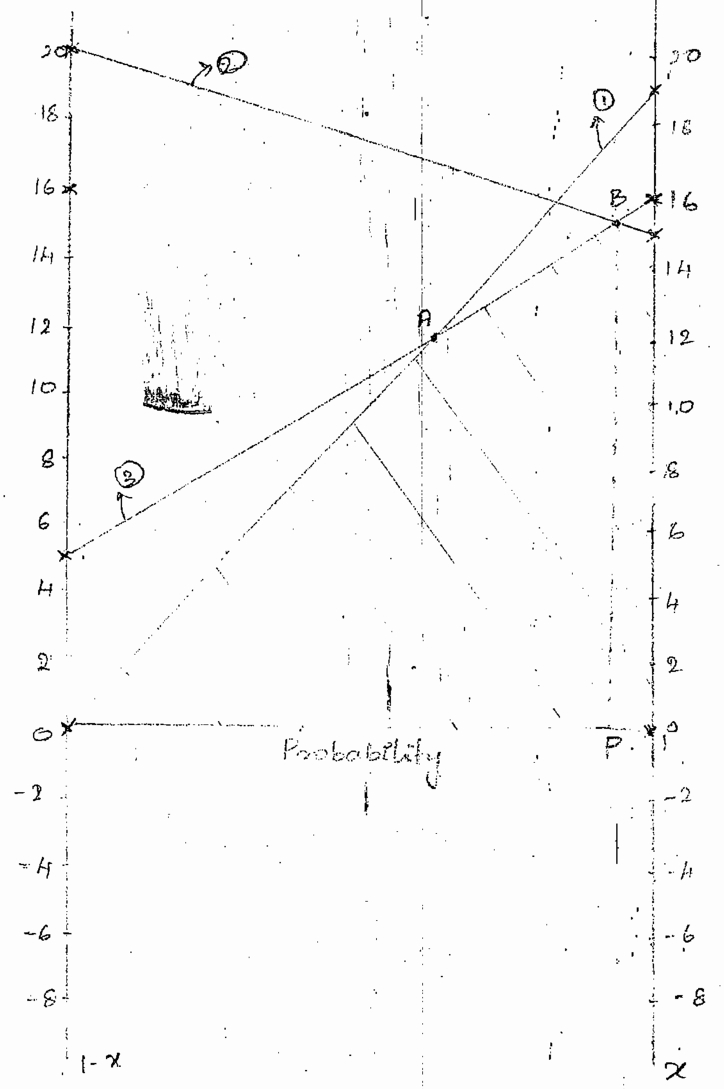
Player B

		a	b	c	d
	1				
Player A	2	19	15	17	16
	3	0	20	15	5

17 > 16, 15 > 5, so eliminate cth column.

	a	b	c	d
1				
2	19	15		16
3	0	20		5

Step 3: Graphical method:



Point B is the intersection of two lines ② and ③.

$$\begin{array}{c|cc} & y & 1-y \\ \hline x & 15 & 16 \\ (1-x) & 20 & 5 \end{array}$$

		Player B	
		y	1-y
Player A	x	15	16
	(1-x)	20	5

For player A:

$$15x+20(1-x) = 16x+5(1-x)$$

$$20-20x+15x = 16x+5-5x.$$

$$20-15x-11x=5$$

$$-16x = 5-20 \Rightarrow x = 15/16$$

$$\boxed{x = \frac{15}{16}} \quad \boxed{1-x = 1/16}$$

$$\text{Value of game} = 15x + 20(1-x)$$

$$= 15 \times \frac{15}{16} + 20 \times \frac{1}{16} = \frac{225+20}{16} = \frac{245}{16}$$

$$\therefore \boxed{\text{Value of game} = \frac{245}{16}}$$

For player B:

$$15y + 16(1-y) = 20y + 5(1-y)$$

$$15y + 16 - 16y = 20y + 5 - 5y$$

$$16 - y = 15y + 5$$

$$16y = 11$$

$$\boxed{y = \frac{11}{16}}$$

$$\boxed{1-y = \frac{5}{16}}$$

$$\text{Value of game} = 15y + 16(1-y)$$

$$= 15 \times \frac{11}{16} + 16 \times \frac{5}{16}$$

$$= \frac{165 + 80}{16}$$

$$= \frac{245}{16}$$

$$\therefore \boxed{\text{Value of game} = \frac{245}{16}}$$

∴ Conclusion:

$$\therefore \boxed{\begin{array}{l} (x_1, x_2, x_3) = (0, 15/16, 1/16) \\ (y_1, y_2, y_3, y_4) = (0, 11/16, 0, 5/16) \end{array}}$$

15.2 DECISION MAKING WITHOUT EXPERIMENTATION

Before seeking a solution to the first Goferbroke Co. problem, we will formulate a general framework for decision making.

In general terms, the decision maker must choose an **alternative** from a set of possible decision alternatives. The set contains all the *feasible alternatives* under consideration for how to proceed with the problem of concern.

This choice of an alternative must be made in the face of uncertainty, because the outcome will be affected by random factors that are outside the control of the decision maker. These random factors determine what situation will be found at the time that the decision alternative is executed. Each of these possible situations is referred to as a possible **state of nature.**

For each combination of a decision alternative and a state of nature, the decision maker knows what the resulting **payoff** would be. The payoff is a quantitative measure of the value to the decision maker of the consequences of the outcome. For example, the payoff frequently is represented by the *net monetary gain* (profit), although other measures also can be used (as described in Sec. 15.6). If the consequences of the outcome do not become completely certain even when the state of nature is given, then the payoff becomes an *expected value* (in the statistical sense) of the measure of the consequences. A **payoff table** commonly is used to provide the payoff for each combination of an action and a state of nature.

If you previously studied game theory (Chap. 14), we should point out an interesting analogy between this decision analysis framework and the two-person, zero-sum games described in Chap. 14. The *decision maker* and *nature* can be viewed as the *two players* of such a game. The *alternatives* and the possible *states of nature* can then be viewed as the available *strategies* for these respective players, where each combination of strategies results in some *payoff* to player 1 (the decision maker). From this viewpoint, the decision analysis framework can be summarized as follows:

1. The *decision maker* needs to choose one of the *decision alternatives.*
2. *Nature* then would choose one of the possible *states of nature.*
3. Each combination of a decision alternative and state of nature would result in a *payoff,* which is given as one of the entries in a *payoff table.*
4. This payoff table should be used to find an *optimal alternative* for the decision maker according to an appropriate criterion.

Soon we will present three possibilities for this criterion, where the first one (the maximin payoff criterion) comes from game theory.

However, this analogy to two-person, zero-sum games breaks down in one important respect. In game theory, *both* players are assumed to be *rational* and choosing their strategies to *promote their own welfare.* This description still fits the decision maker, but certainly not nature. By contrast, nature now is a passive player that chooses its strategies (states of nature) in some random fashion. This change means that the game theory criterion for how to choose an optimal strategy (alternative) will not appeal to many decision makers in the current context.

One additional element needs to be added to the decision analysis framework. The decision maker generally will have some information that should be taken into account about the relative likelihood of the possible states of nature. Such information can usually be translated to a probability distribution, acting as

TABLE 15.2 Payoff table for the decision analysis formulation of the Goferbroke Co. problem

	State of Nature	
Alternative	**Oil**	**Dry**
1. Drill for oil	700	−100
2. Sell the land	90	90
Prior probability	0.25	0.75

of an individual. The probabilities for the respective states of nature provided by the prior distribution are called **prior probabilities**[1].

Formulation of the Prototype Example in This Framework

As indicated in Table 15.1, the Goferbroke Co. has two possible decision alternatives under consideration: drill for oil or sell the land. The possible states of nature are that the land contains oil and that it does not, as designated in the column headings of Table 15.1 by *oil* and *dry*. Since the consulting geologist has estimated that there is 1 chance in 4 of oil (and so 3 chances in 4 of no oil), the prior probabilities of the two states of nature are 0.25 and 0.75, respectively. Therefore, with the payoff in units of thousands of dollars of profit, the payoff table can be obtained directly from Table 15.1, as shown in Table 15.2.

We will use this payoff table next to find the optimal alternative according to each of the three criteria described below.

The Maximin Payoff Criterion

If the decision maker's problem were to be viewed as a *game against nature*, then game theory would say to choose the decision alternative according to the *minimax criterion* (as described in Sec. 14.2). From the viewpoint of player 1 (the decision maker), this criterion is more aptly named the *maximin payoff criterion*, as summarized below.

Maximin payoff criterion: For each possible decision alternative, find the *minimum payoff* over all possible states of nature. Next, find the *maximum* of these minimum payoffs. Choose the alternative whose minimum payoff gives this maximum.

Table 15.3 shows the application of this criterion to the prototype example. Thus, since the minimum payoff for selling (90) is larger than that for drilling (−100), the former alternative (sell the land) will be chosen.

The rationale for this criterion is that it provides the *best guarantee* of the payoff that will be obtained. Regardless of what the true state of nature turns out to be for the example, the payoff from selling the land cannot be less than 90, which provides the best available guarantee. Thus, this criterion takes the pessimistic viewpoint that, regardless of which alternative is selected, the worst state of nature for that alternative is likely to occur, so we should choose the alternative which provides the best payoff with its worst state of nature.

TABLE 15.3 Application of the maximin payoff criterion to the first Goferbroke Co. problem

	State of Nature			
Alternative	**Oil**	**Dry**	**Minimum**	
1. Drill for oil	700	−100	−100	
2. Sell the land	90	90	90	← Maximin value
Prior probability	0.25	0.75		

TABLE 15.4 Application of the maximum likelihood criterion to the first Goferbroke Co. problem

	State of Nature			
Alternative	**Oil**	**Dry**		
1. Drill for oil	700	−100	−100	
2. Sell the land	90	90	90	← Maximum in this column
Prior probability	0.25	0.75		
		↑ Maximum		

This rationale is quite valid when one is competing against a rational and malevolent opponent. However, this criterion is not often used in games against nature because it is an extremely conservative criterion in this context. In effect, it assumes that nature is a conscious opponent that wants to inflict as much damage as possible on the decision maker. Nature is not a malevolent opponent, and the decision maker does not need to focus solely on the worst possible payoff from each alternative. This is especially true when the worst possible payoff from an alternative comes from a relatively unlikely state of nature.

Thus, this criterion normally is of interest only to a very cautious decision maker.

The Maximum Likelihood Criterion

The next criterion focuses on the *most likely* state of nature, as summarized below.

> **Maximum likelihood criterion:** Identify the most likely state of nature (the one with the largest prior probability). For this state of nature, find the decision alternative with the maximum payoff. Choose this decision alternative.

Applying this criterion to the example, Table 15.4 indicates that the *Dry* state has the largest prior probability. In the Dry column, the sell alternative has the maximum payoff, so the choice is to sell the land.

The appeal of this criterion is that the most important state of nature is the most likely one, so the alternative chosen is the best one for this particularly important state of nature. Basing the decision on the assumption that this state of nature will occur tends to give a better chance of a favorable outcome than assuming any other state of nature. Furthermore, the criterion does not rely on questionable subjective estimates of the probabilities of the respective states of nature other than identifying the most likely state.

The major drawback of the criterion is that it completely ignores much relevant information. No state of nature is considered other than the most likely one. In a problem with many possible states of nature the probability of the most likely one may be quite small, so focusing on just this one state of nature is quite unwarranted. Even in the example, where the prior probability of the *Dry* state is 0.75, this criterion ignores the extremely attractive payoff of 700 if the company drills and finds oil. In effect, the criterion does not permit gambling on a low-probability big payoff, no matter how attractive the gamble may be.

Bayes' Decision Rule[1]

Our third criterion, and the one commonly chosen, is *Bayes' decision rule,* described below.

> **Bayes' decision rule:** Using the best available estimates of the probabilities of the respective states of nature (currently the prior probabilities), calculate the expected value of the payoff for each of the possible decision alternatives. Choose the decision alternative with the maximum expected payoff.

For the prototype example, these expected payoffs are calculated directly from Table 15.2 as follows:

$$\begin{aligned} E[\text{Payoff (drill)}] &= 0.25(700) + 0.75(-100) \\ &= 100. \\ E[\text{Payoff (sell)}] &= 0.25(90) + 0.75(90) \\ &= 90. \end{aligned}$$

Since 100 is larger than 90, the alternative selected is to drill for oil.

Note that this choice contrasts with the selection of the sell alternative under each of the two preceding criteria.

The big advantage of Bayes' decision rule is that it incorporates all the available information, including all the payoffs and the best available estimates of the probabilities of the respective states of nature.

It is sometimes argued that these estimates of the probabilities necessarily are largely subjective and so are too shaky to be trusted. There is no accurate way of predicting the future, including a future state of nature, even in probability terms. This argument has some validity. The reasonableness of the estimates of the probabilities should be assessed in each individual situation.

Nevertheless, under many circumstances, past experience and current evidence enable one to develop reasonable estimates of the probabilities. Using this information should provide better grounds for a sound decision than ignoring it. Furthermore, experimentation frequently can be conducted to improve these estimates, as described in the next section. Therefore, we will be using only Bayes' decision rule throughout the remainder of the chapter.

To assess the effect of possible inaccuracies in the prior probabilities, it often is helpful to conduct sensitivity analysis, as described below.

Sensitivity Analysis with Bayes' Decision Rule

Sensitivity analysis commonly is used with various applications of operations research to study the effect if some of the numbers included in the mathematical model are not correct. In this case, the mathematical model is represented by the payoff table shown in Table 15.2. The numbers in this table that are most questionable are the prior probabilities. We will focus the sensitivity analysis on these numbers, although a similar approach could be applied to the payoffs given in the table.

The sum of the two prior probabilities must equal 1, so increasing one of these probabilities automatically decreases the other one by the same amount, and vice versa. Goferbroke's management feels that the true chances of having oil on the tract of land are likely to lie somewhere between 15 and 35 percent. In other words, the true prior probability of having oil is likely to be in the range from 0.15 to 0.35, so the corresponding prior probability of the land being dry would range from 0.85 to 0.65.

Letting

p = prior probability of oil,

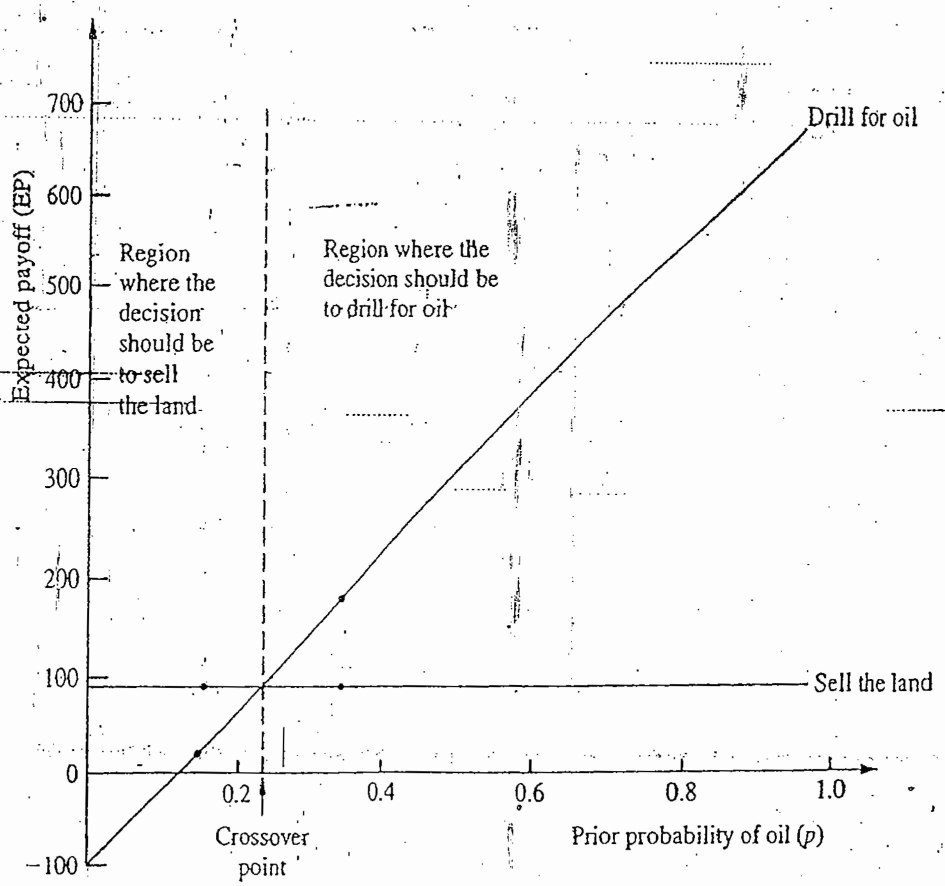
the expected payoff from drilling for any p is

$$
\begin{aligned}
E[\text{Payoff (drill)}] &= 700p - 100(1-p)\\
&= 800p - 100.
\end{aligned}
$$

The slanting line in Fig. 15.1 shows the plot of this expected payoff versus p. Since the payoff from selling the land would be 90 for any p, the flat line in Fig. 15.1 gives E[Payoff (sell)] versus p.

The four dots in Fig. 15.1 show the expected payoff for the two decision alternatives when $p = 0.15$ or $p = 0.35$. When $p = 0.15$, the decision swings over to selling the land by a wide margin (an expected payoff of 90 versus only 20 for drilling). However, when $p = 0.35$, the decision is to drill by a wide margin (expected payoff = 180 versus only 90 for selling). Thus, the decision is very *sensitive* to p. This sensitivity analysis has revealed that it is important to do more, if possible, to develop a more precise estimate of the true value of p.

FIGURE 15.1
Graphical display of how the expected payoff for each decision alternative changes when the prior probability of oil changes for the first Goferbroke Co. problem.



The point in Fig. 15.1 where the two lines intersect is the **crossover point** where the decision shifts from one alternative (sell the land) to the other (drill for oil) as the prior probability increases. To find this point, we set

$$E[\text{Payoff (drill)}] = E[\text{Payoff (sell)}]$$

$$800p - 100 = 90$$

$$p = \frac{190}{800} = 0.2375$$

Conclusion: Should sell the land if $p < 0.2375$.
Should drill for oil if $p > 0.2375$.

Thus, when trying to refine the estimate of the true value of p, the key question is whether it is smaller or larger than 0.2375.

For other problems that have more than two decision alternatives, the same kind of analysis can be applied. The main difference is that there now would be more than two lines (one per alternative) in the graphical display corresponding to Fig. 15.1. However, the top line for any particular value of the prior probability still indicates which alternative should be chosen. With more than two lines, there might be more than one crossover point where the decision shifts from one alternative to another.

You can see an example of performing this kind of analysis with three decision alternatives in the Worked Examples section of the CD-ROM. (This same example also illustrates the application of all three decision criteria considered in this section.)

For a problem with more than two possible states of nature, the most straightforward approach is to focus the sensitivity analysis on only two states at a time as described above. This again would involve investigating what happens when the prior probability of one state increases as the prior probability of the other state decreases by the same amount, holding fixed the prior probabilities of the remaining states. This procedure then can be repeated for as many other pairs of states as desired.

Practitioners sometimes use software to assist them in performing this kind of sensitivity analysis, including generating the graphs. For example, an Excel add-in in your OR Courseware called *SensIt* is designed specifically for conducting sensitivity analysis with probabilistic models such as when applying Bayes' decision rule. Complete documentation for SensIt is included on your CD-ROM. Section 15.5 will describe and illustrate the application of SensIt.

Because the decision the Goferbroke Co. should make depends so critically on the true probability of oil, serious consideration should be given to conducting a seismic survey to estimate this probability more closely. We will explore this option in the next two sections.

15.3 DECISION MAKING WITH EXPERIMENTATION

Frequently, additional testing (experimentation) can be done to improve the preliminary estimates of the probabilities of the respective states of nature provided by the prior probabilities. These improved estimates are called **posterior probabilities.**

We first update the Goferbroke Co. example to incorporate experimentation, then describe how to derive the posterior probabilities, and finally discuss how to decide whether it is worthwhile to conduct experimentation.

Continuing the Prototype Example

As mentioned at the end of Sec. 15.1, an available option before making a decision is to conduct a detailed seismic survey of the land to obtain a better estimate of the probability of oil. The cost is $30,000.

A seismic survey obtains seismic soundings that indicate whether the geological structure is favorable to the presence of oil. We will divide the possible findings of the survey into the following two categories:

USS: Unfavorable seismic soundings; oil is fairly unlikely.
FSS: Favorable seismic soundings; oil is fairly likely.

Based on past experience, if there is oil, then the probability of unfavorable seismic soundings is

$$P(\text{USS} \mid \text{State} = \text{Oil}) = 0.4, \quad \text{so} \quad P(\text{FSS} \mid \text{State} = \text{Oil}) = 1 - 0.4 = 0.6.$$

Similarly, if there is no oil (i.e., the true state of nature is *Dry*), then the probability of unfavorable seismic soundings is estimated to be

$$P(\text{USS} \mid \text{State} = \text{Dry}) = 0.8, \quad \text{so} \quad P(\text{FSS} \mid \text{State} = \text{Dry}) = 1 - 0.8 = 0.2.$$

We soon will use these data to find the posterior probabilities of the respective states of nature *given* the seismic soundings.

Posterior Probabilities

Proceeding now in general terms, we let

n = number of possible states of nature;
$P(\text{State} = \text{state } i)$ = prior probability that true state of nature is state i, for $i = 1, 2, \dots, n$;
Finding = finding from experimentation (a random variable);
Finding j = one possible value of finding;
$P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j)$ = posterior probability that true state of nature is state i, given that Finding = finding j, for $i = 1, 2, \dots, n$.

The question currently being addressed is the following:

Given $P(\text{State} = \text{state } i)$ and $P(\text{Finding} = \text{finding } j \mid \text{State} = \text{state } i)$, for $i = 1, 2, \dots, n$, what is $P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j)$?

This question is answered by combining the following standard formulas of probability theory:

$$P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j) = \frac{P(\text{State} = \text{state } i, \text{ Finding} = \text{finding } j)}{P(\text{Finding} = \text{finding } j)}$$

$$P(\text{Finding} = \text{finding } j) = \sum_{k=1}^{n} P(\text{State} = \text{state } k, \text{ Finding} = \text{finding } j)$$

$$P(\text{State} = \text{state } i, \text{ Finding} = \text{finding } j) = P(\text{Finding} = \text{finding } j \mid \text{State} = \text{state } i) \, P(\text{State} = \text{state } i).$$

Therefore, for each $i = 1, 2, \dots, n$, the desired formula for the corresponding posterior probability is

$$P(\text{State} = \text{state } i \mid \text{Finding} = \text{finding } j) = \frac{P(\text{Finding} = \text{finding } j \mid \text{State} = \text{state } i) P(\text{State} = \text{state } i)}{\sum_{k=1}^{n} P(\text{Finding} = \text{finding } j \mid \text{State} = \text{state } k) P(\text{State} = \text{state } k)}$$

(This formula often is referred to as **Bayes' theorem** because it was developed by Thomas Bayes, the same 18th-century mathematician who is credited with developing Bayes' decision rule.)

Now let us return to the prototype example and apply this formula. If the finding of the seismic survey is unfavorable seismic soundings (USS), then the posterior probabilities are

$$P(\text{State} = \text{Oil} \mid \text{Finding} = \text{USS}) = \frac{0.4(0.25)}{0.4(0.25) + 0.8(0.75)} = \frac{1}{7},$$

$$P(\text{State} = \text{Dry} \mid \text{Finding} = \text{USS}) = 1 - \frac{1}{7} = \frac{6}{7}.$$

Similarly, if the seismic survey gives favorable seismic soundings (FSS), then

$$P(\text{State} = \text{Oil} \mid \text{Finding} = \text{FSS}) = \frac{0.6(0.25)}{0.6(0.25) + 0.2(0.75)} = \frac{1}{2},$$

$$P(\text{State} = \text{Dry} \mid \text{Finding} = \text{FSS}) = 1 - \frac{1}{2} = \frac{1}{2}.$$

The **probability tree diagram** in Fig. 15.2 shows a nice way of organizing these calculations in an intuitive manner. The prior probabilities in the first column and the conditional probabilities in the second column are part of the input data for the problem. Multiplying each probability in the first column by a probability in the second column gives the corresponding joint probability in the third column. Each joint probability then becomes the numerator in the calculation of the corresponding posterior probability in the

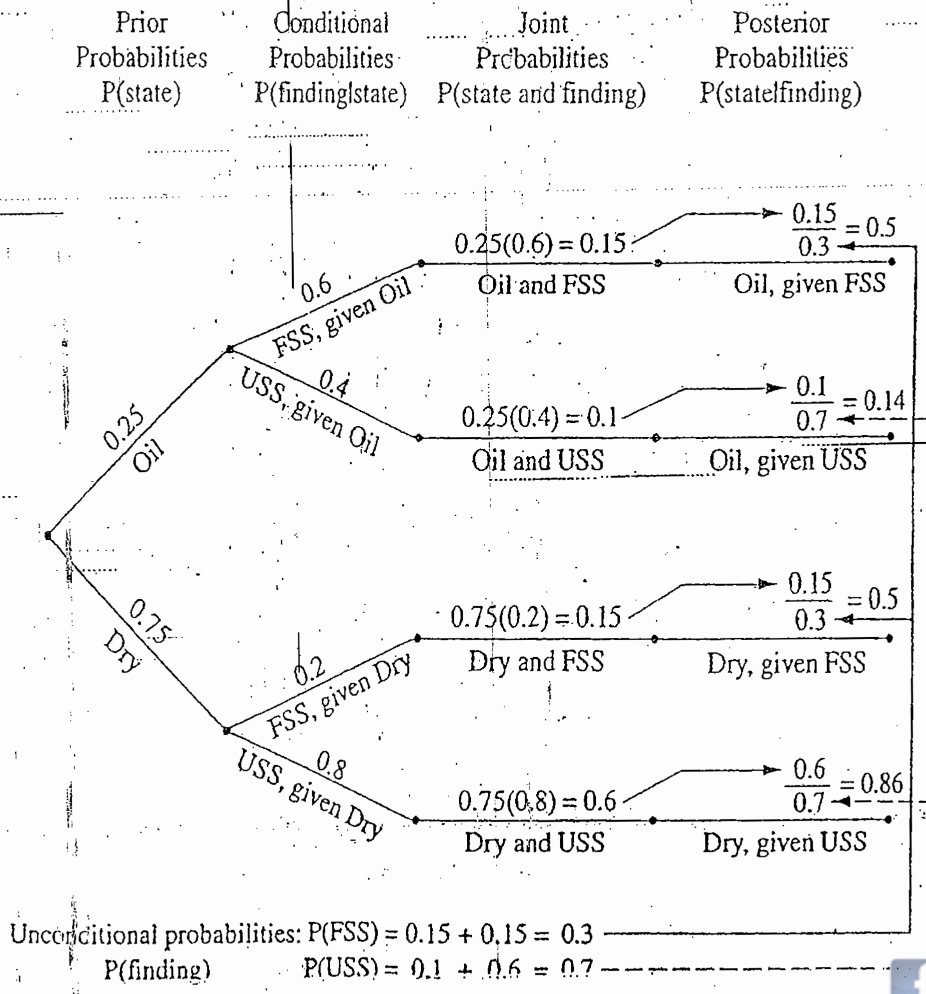


FIGURE 15.2
Probability tree diagram for the full Goferbroke Co. problem showing all the probabilities leading to the calculation of each posterior probability of the state of nature given the finding of the seismic survey.

fourth column. Cumulating the joint probabilities with the same finding (as shown at the bottom of the figure) provides the denominator for each posterior probability with this finding. (If you would like to see another example of using a probability tree diagram to determine the posterior probabilities, one is included in the Worked Examples section of the CD-ROM.)

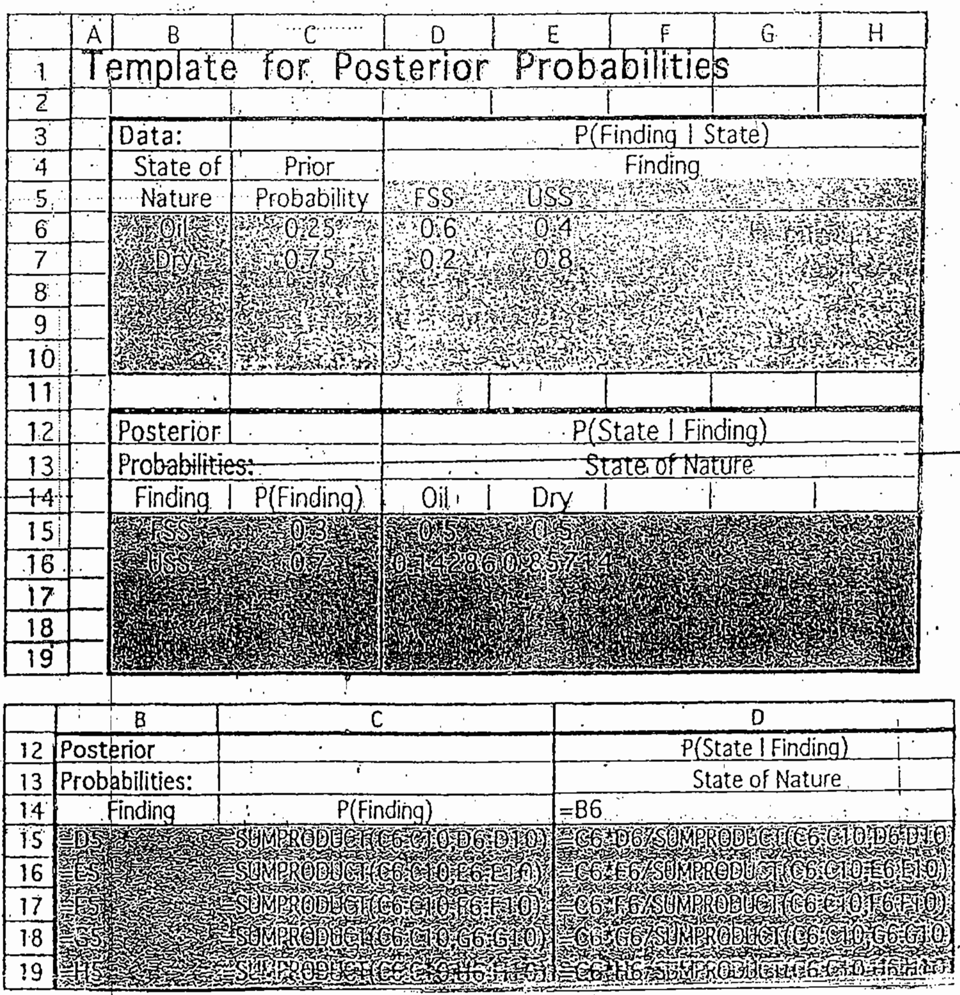
Your OR Courseware also includes an Excel template for computing these posterior probabilities, as shown in Fig. 15.3.

After these computations have been completed, Bayes' decision rule can be applied just as before, with the posterior probabilities now replacing the prior probabilities. Again, by using the payoffs (in units of thousands of dollars) from Table 15.2 and subtracting the cost of the experimentation, we obtain the results shown below.

Expected payoffs if finding is unfavorable seismic soundings (USS):

$$E[\text{Payoff (drill} \mid \text{Finding} = \text{USS)}] = \frac{1}{7}(700) + \frac{6}{7}(-100) - 30$$
$$= -15.7.$$

$$E[\text{Payoff (sell} \mid \text{Finding} = \text{USS)}] = \frac{1}{7}(90) + \frac{6}{7}(90) - 30$$
$$= 60.$$



	A	B	C	D	E	F	G	H
1		Template for Posterior Probabilities						
2								
3		Data:		P(Finding | State)				
4		State of	Prior	Finding				
5		Nature	Probability	FSS	USS			
6		Oil	0.25	0.6	0.4			
7		Dry	0.75	0.2	0.8			
8								
9								
10								
11								
12		Posterior		P(State | Finding)				
13		Probabilities:		State of Nature				
14		Finding	P(Finding)	Oil	Dry			
15		FSS	0.3	0.5	0.5			
16		USS	0.7	0.14286	0.85714			
17								
18								
19								

	B	C	D
12	Posterior		P(State | Finding)
13	Probabilities:		State of Nature
14	Finding	P(Finding)	=B6
15	=D5	=SUMPRODUCT(C6:C10,D6:D10)	=C6*D6/SUMPRODUCT(C6:C10,D6:D10)
16	=E5	=SUMPRODUCT(C6:C10,E6:E10)	=C6*E6/SUMPRODUCT(C6:C10,E6:E10)
17	=F5	=SUMPRODUCT(C6:C10,F6:F10)	=C6*F6/SUMPRODUCT(C6:C10,F6:F10)
18	=G5	=SUMPRODUCT(C6:C10,G6:G10)	=C6*G6/SUMPRODUCT(C6:C10,G6:G10)
19	=H5	=SUMPRODUCT(C6:C10,H6:H10)	=C6*H6/SUMPRODUCT(C6:C10,H6:H10)

FIGURE 15.3 This posterior probabilities template in your OR Courseware enables efficient calculation of posterior probabilities, as illustrated here for the full Goferbroke Co. problem.

Expected payoffs if finding is favorable seismic soundings (FSS):

$$E[\text{Payoff (drill} \mid \text{Finding} = \text{FSS})] = \frac{1}{2}(700) + \frac{1}{2}(-100) - 30$$
$$= 270.$$
$$E[\text{Payoff (sell} \mid \text{Finding} = \text{FSS})] = \frac{1}{2}(90) + \frac{1}{2}(90) - 30$$
$$= 60.$$

Since the objective is to maximize the expected payoff, these results yield the optimal policy shown in Table 15.5.

However, what this analysis does not answer is whether it is worth spending $30,000 to conduct the experimentation (the seismic survey). Perhaps it would be better to forgo this major expense and just use the optimal solution without experimentation (drill for oil, with an expected payoff of $100,000). We address this issue next.

The Value of Experimentation

Before performing any experiment, we should determine its potential value. We present two complementary methods of evaluating its potential value.

The first method assumes (unrealistically) that the experiment will remove *all* uncertainty about what the true state of nature is, and then this method makes a very quick calculation of what the resulting *improvement in the expected payoff* would be (ignoring the cost of the experiment). This quantity, called the *expected value of perfect information*, provides an *upper bound* on the potential value of the experiment. Therefore, if this upper bound is less than the cost of the experiment, the experiment definitely should be forgone.

However, if this upper bound exceeds the cost of the experiment, then the second (slower) method should be used next. This method calculates the *actual* improvement in the expected payoff (ignoring the cost of the experiment) that would result from performing the experiment. Comparing this improvement with the cost indicates whether the experiment should be performed.

Expected Value of Perfect Information. Suppose now that the experiment could definitely identify what the true state of nature is, thereby providing "perfect" information. Whichever state of nature is identified, you naturally choose the action with the maximum payoff for that state. We do not know in advance which state of nature will be identified, so a calculation of the expected payoff with perfect information (ignoring the cost of the experiment) requires weighting the maximum payoff for each state of nature by the prior probability of that state of nature.

This calculation is shown at the bottom of Table 15.6 for the full Goferbroke Co. problem, where the expected value of perfect information is 242.5.

TABLE 15.5 The optimal policy with experimentation, under Bayes' decision rule, for the full Goferbroke Co. problem

Finding from Seismic Survey	Optimal Alternative	Expected Payoff Excluding Cost of Survey	Expected Payoff Including Cost of Survey
USS	Sell the land	90	60
FSS	Drill for oil	300	270

TABLE 15.6 Expected payoff with perfect information for the full Goferbroke Co. problem

	State of Nature	
Alternative	**Oil**	**Dry**
1. Drill for oil	700	−100
2. Sell the land	90	90
Maximum payoff	700	90
Prior probability	0.25	0.75

Expected payoff with perfect information = 0.25(700) + 0.75(90) = 242.5

Thus, if the Goferbroke Co. could learn before choosing its action whether the land contains oil, the expected payoff as of now (before acquiring this information) would be $242,500 (excluding the cost of the experiment generating the information.)

To evaluate whether the experiment should be conducted, we now use this quantity to calculate the expected value of perfect information.

> The **expected value of perfect information,** abbreviated EVPI, is calculated as
>
> EVPI = expected payoff with perfect information − expected payoff without experimentation.[1]
>
> Thus, since experimentation usually cannot provide perfect information, EVPI provides an upper bound on the expected value of experimentation.

For this same example, we found in Sec. 15.2 that the expected payoff without experimentation (under Bayes' decision rule) is 100. Therefore,

$$\text{EVPI} = 242.5 - 100 = 142.5.$$

Since 142.5 far exceeds 30, the cost of experimentation (a seismic survey), it may be worthwhile to proceed with the seismic survey. To find out for sure, we now go to the second method of evaluating the potential benefit of experimentation.

Expected Value of Experimentation. Rather than just obtain an upper bound on the *expected increase in payoff* (excluding the cost of the experiment) due to performing experimentation, we now will do somewhat more work to calculate this expected increase directly. This quantity is called the *expected value of experimentation.*

Calculating this quantity requires first computing the expected payoff with experimentation (excluding the cost of the experiment). Obtaining this latter quantity requires doing all the work described earlier to find all the posterior probabilities, the resulting optimal policy with experimentation, and the corresponding expected payoff (excluding the cost of the experiment) for each possible finding from the experiment. Then each of these expected payoffs needs to be weighted by the probability of the corresponding finding, that is,

$$\text{Expected payoff with experimentation} = \sum_j P(\text{Finding} = \text{finding } j) \; E[\text{payoff} \mid \text{Finding} = \text{finding } j],$$

where the summation is taken over all possible values of j.

For the prototype example, we have already done all the work to obtain the terms on the right side of this equation. The values of $P(\text{Finding} = \text{finding } j)$ for the two possible findings from the seismic survey—unfavorable (USS) and favorable (FSS)—were calculated at the bottom of the probability tree diagram in Fig. 15.2 as

$$P(\text{USS}) = 0.7, \qquad P(\text{FSS}) = 0.3.$$

For the optimal policy with experimentation, the corresponding expected payoff (excluding the cost of the seismic survey) for each finding was obtained in the third column of Table 15.5 as

$$E(\text{Payoff} \mid \text{Finding} = \text{USS}) = 90,$$
$$E(\text{Payoff} \mid \text{Finding} = \text{FSS}) = 270.$$

With these numbers,

$$\begin{aligned}\text{Expected payoff with experimentation} &= 0.7(90) + 0.3(300)\\ &= 153.\end{aligned}$$

Now we are ready to calculate the expected value of experimentation.

The expected value of experimentation, abbreviated **EVE**, is calculated as

EVE = expected payoff with experimentation − expected payoff without experimentation.

Thus, EVE identifies the potential value of experimentation.

For the Goferbroke Co.,

$$\text{EVE} = 153 - 100 = 53.$$

Since this value exceeds 30, the cost of conducting a detailed seismic survey (in units of thousands of dollars), this experimentation should be done.

5.4 DECISION TREES

ecision trees provide a useful way of *visually displaying* the problem and then *organizing the computa-* *onal work* already described in the preceding two sections. These trees are especially helpful when a *se-* *uence of decisions* must be made.

onstructing the Decision Tree

The prototype example involves a sequence of two decisions:

1. Should a seismic survey be conducted before an action is chosen?
2. Which action (drill for oil or sell the land) should be chosen?

The corresponding decision tree (before adding numbers and performing computations) is displayed in Fig. 15.4.

The junction points in the decision tree are referred to as **nodes** (or forks), and the lines are called **ranches.**

A **decision node**, represented by a square, indicates that a decision needs to be made at that point in the process. An **event node** (or chance node), represented by a circle, indicates that a random event occurs at that point.

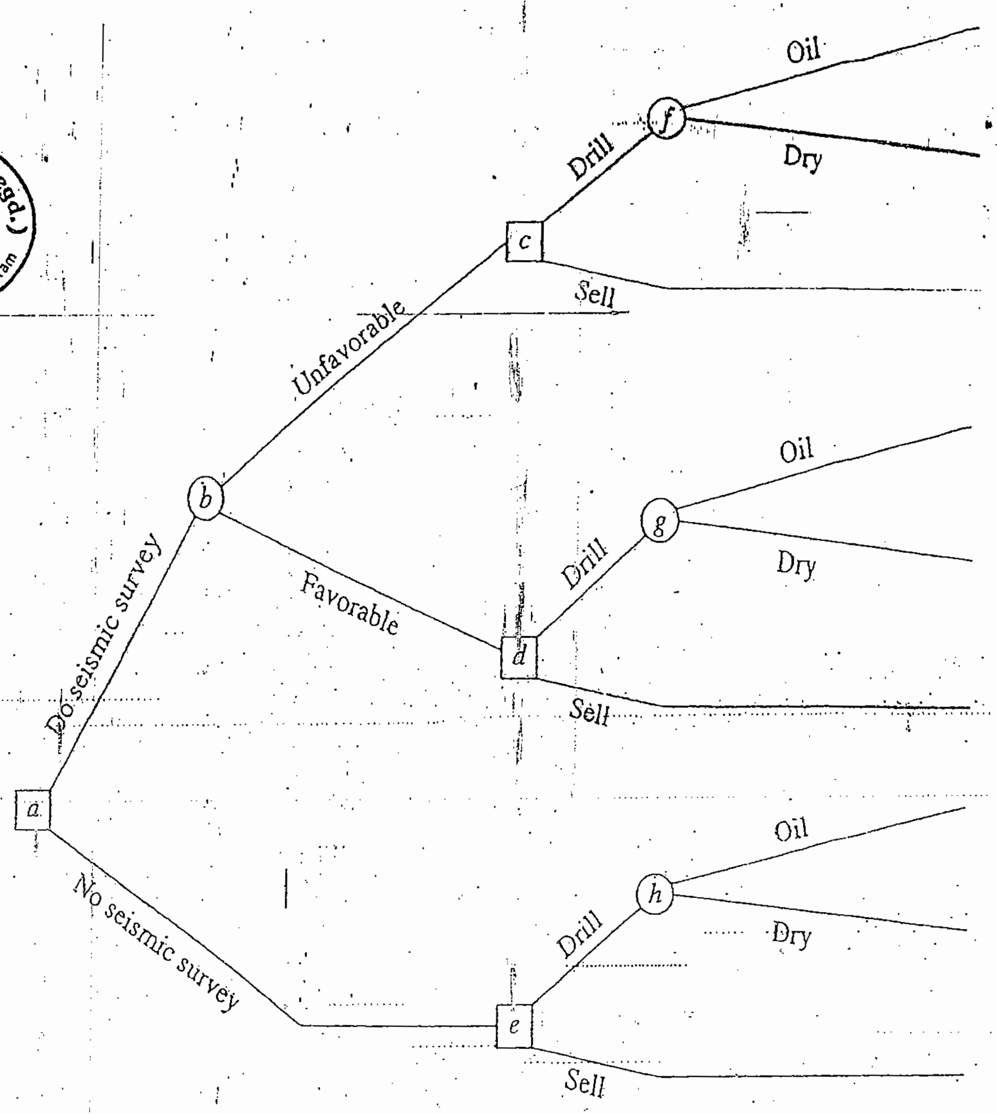


FIGURE 15.4
The decision tree (before including any numbers) for the full Goferbroke Co. problem.

Thus, in Fig. 15.4, the first decision is represented by decision node *a*. Node *b* is an event node representing the random event of the outcome of the seismic survey. The two branches emanating from event node *b* represent the two possible outcomes of the survey. Next comes the second decision (nodes *c*, *d*, and *e*) with its two possible choices. If the decision is to drill for oil, then we come to another event node (nodes *f*, *g*, and *h*), where its two branches correspond to the two possible states of nature.

Note that the path followed from node *a* to reach any terminal branch (except the bottom one) is determined both by the decisions made and by random events that are outside the control of the decision maker. This is characteristic of problems addressed by decision analysis.

The next step in constructing the decision tree is to insert numbers into the tree as shown in Fig. 15.5. The numbers under or over the branches that are *not* in parentheses are the cash flows (in thousands of dollars) that occur at those branches. For each path through the tree from node *a* to a terminal branch, these same numbers then are added to obtain the resulting total payoff shown in boldface to the right of that branch. The last set of numbers is the probabilities of random events. In particular, since each branch emanating from an event node represents a possible random event, the probability of this event occurring from this node has been inserted in parentheses along this branch. From event node *h*, the probabilities are the prior probabilities of these states of nature, since no seismic survey has been conducted to obtain more information in this case. However, event nodes *f* and *g* lead out of a decision to do the seismic survey (and

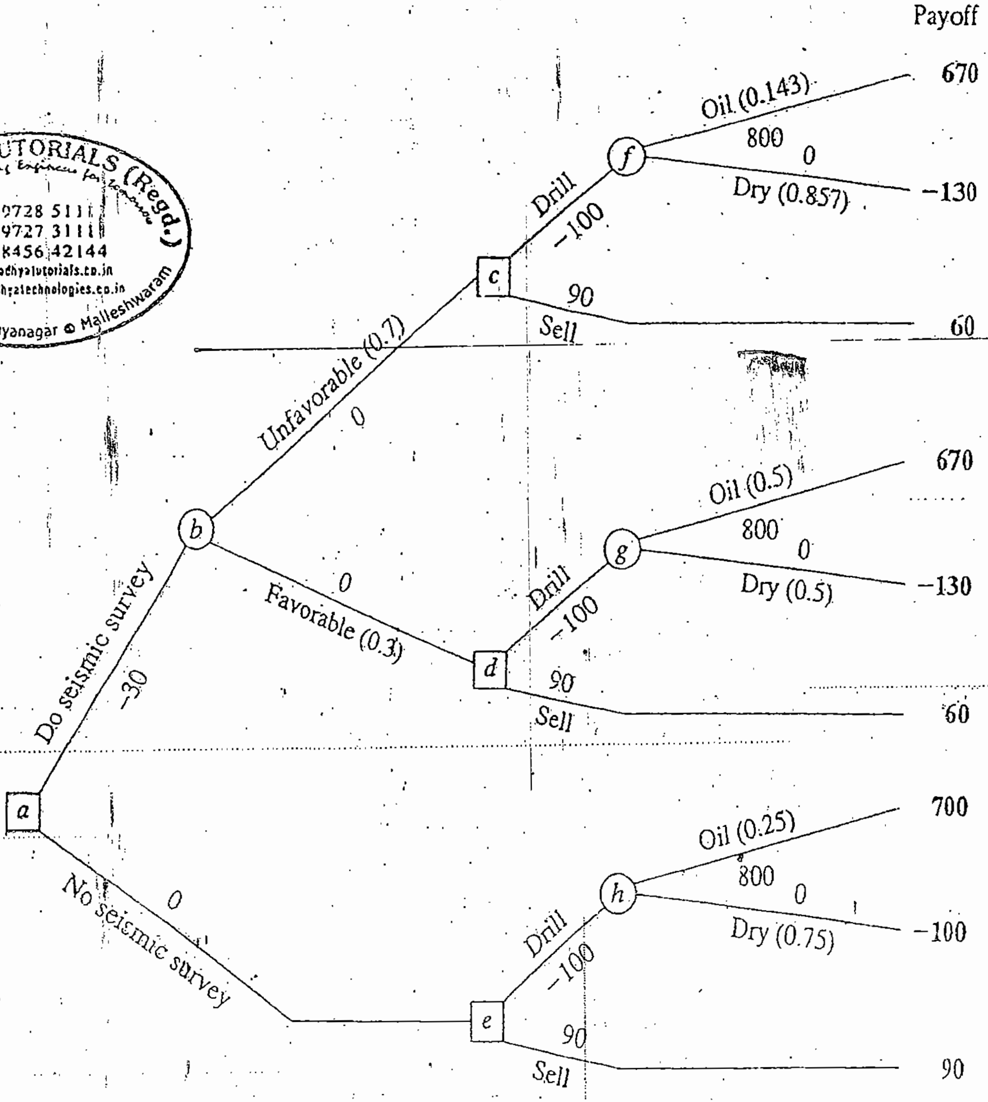


FIGURE 15.5
The decision tree in ig. 15.4 after dding both the robabilities of andom events and e payoffs.

en to drill). Therefore, the probabilities from these event nodes are the *posterior probabilities* of the states nature, given the finding from the seismic survey, where these numbers are given in Figs. 15.2 and 15.3. Fially, we have ~~the two branches emanating from event node *b*. The numbers here are the probabilities of~~ ese findings from the seismic survey, Favorable (FSS) or Unfavorable (USS), as given underneath the obability tree diagram in Fig. 15.2 or in cells C15:C16 of Fig. 15.3.

erforming the Analysis

aving constructed the decision tree, including its numbers, we now are ready to analyze the problem by sing the following procedure.

. Start at the right side of the decision tree and move left one column at a time. For each column, perform either step 2 or step 3 depending upon whether the nodes in that column are event nodes or decision nodes.

For each event node, calculate its *expected payoff* by multiplying the expected payoff of each branch (shown in boldface to the right of the branch) by the probability of that branch and then summing these products. Record this expected payoff for each decision node in boldface next to the node, and designate

3'

this quantity as also being the expected payoff for the branch leading to this node.

3. For each decision node, compare the expected payoffs of its branches and choose the alternative whose branch has the largest expected payoff. In each case, record the choice on the decision tree by inserting a double dash as a barrier through each rejected branch.

To begin the procedure, consider the rightmost column of nodes, namely, event nodes *f*, *g*, and *h*. Applying step 2, their expected payoffs (EP) are calculated as

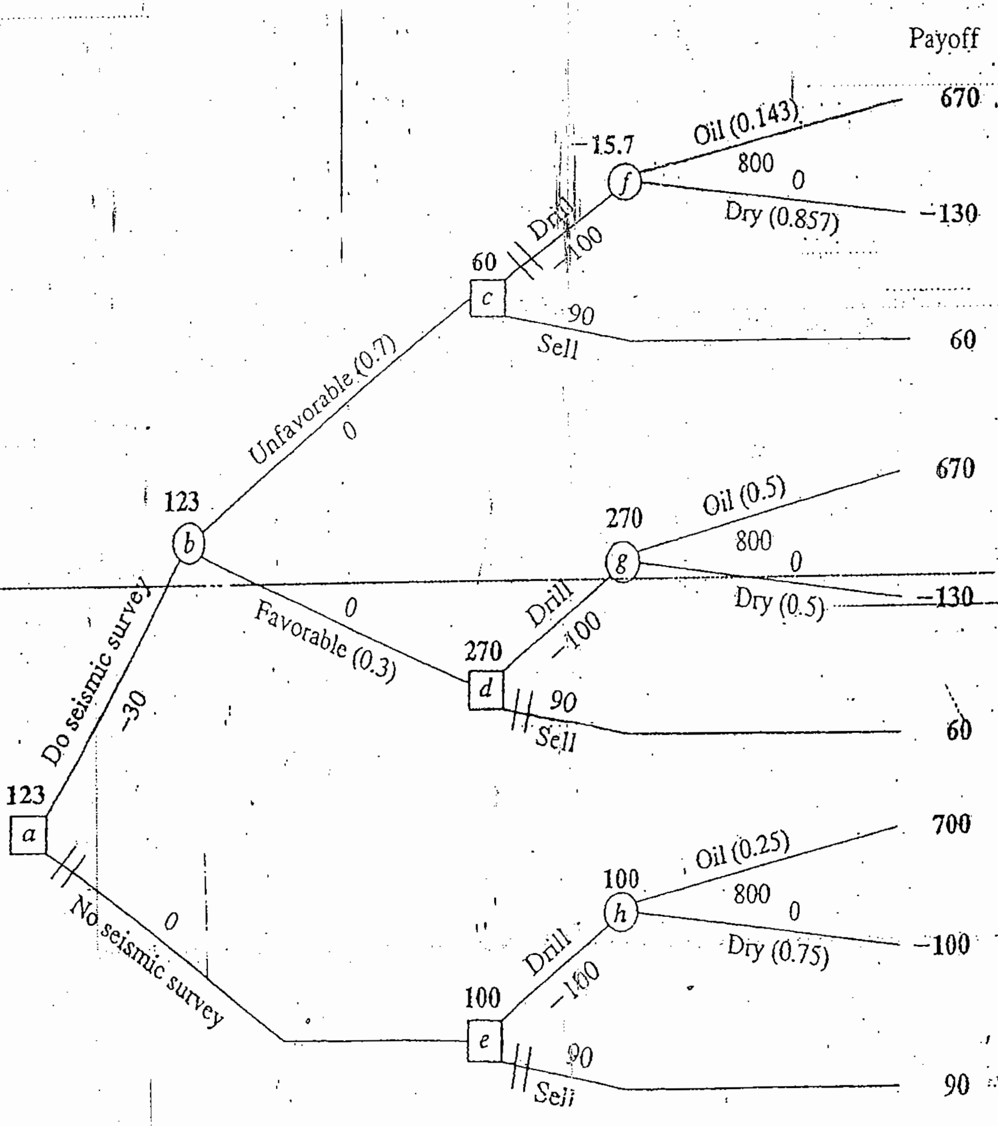
$$\text{EP} = \frac{1}{7}(670) + \frac{6}{7}(-130) = -15.7, \qquad \text{for node } f,$$

$$\text{EP} = \frac{1}{2}(670) + \frac{1}{2}(-130) = 270, \qquad \text{for node } g,$$

$$\text{EP} = \frac{1}{4}(700) + \frac{3}{4}(-100) = 100, \qquad \text{for node } h.$$

These expected payoffs then are placed above these nodes, as shown in Fig. 15.6.

FIGURE 15.6
The final decision tree that records the analysis for the full Goferbroke Co. problem when using monetary payoffs.



Divya

UNIT 7

GAME THEORY, DECISION ANALYSIS

Decision Analysis

It provides a rational approach to the managers in dealing with problems confronted in partial imperfect or uncertain future conditions.

Steps in Decision Theory

Step 1: List all the viable alternatives

Step 2: Identify the expected future events

Step 3: Construct a pay-off table

Step 4: Select optimum decision criterion.

Calculation of EMV (Expected Monetary Value)

1) Construct a conditional pay-off table listing the alternative decisions and the various states of nature.

2) Calculate the EMV for each decision alternative by multiplying the conditional profits by assigned probabilities and adding the resulting conditional values.

3) Select the alternative that yields highest EMV.

Calculation of EOL (Expected Opportunity Loss)

1) Prepare the conditional profit table for each decision event combination and write the associated probabilities.

2) For each event, determine COL (Conditional Opportunity Loss) by subtracting the pay-off from the maximum pay-off for that event.

3) Calculate the EOL for each decision alternative by multiplying the COL's by the associated probabilities and adding the values.

4) Select the alternative that yields the lowest EOL.

Decision Trees

It is a graphical representation of the decision process indicating the decision alternatives, states of nature, probabilities attached to the states of nature and conditional benefits and losses. It consists of network of nodes and branches. Two types of nodes are used: decision node represented by a square and state of nature node represented by a circle. Alternative courses of action originate from the decision node as main branches. At the end of each decision branch, there is a state of nature node from which emanate chance events in the form of sub-events (branches).

Steps in Decision Tree Analysis

1. Identify the decision points and the alternative courses of action at each decision point systematically.
2. At each point, determine the probability and the pay-off associated with each course of action.
3. Commencing from the extreme right end, compute EMV for each course of action.
4. Choose the course of action that yields the best pay-off for each of the decisions.
5. Proceed backwards to the next stage of decision point.
6. Repeat the above steps till the first decision point is reached.

7. Finally identify the courses of action to be adopted from the beginning to the end under different possible outcomes for the situations as a whole.

Advantages and Limitations of Decision Tree Approach

Advantages:

1. It structures the decision process and helps decision-making in an orderly systematic and sequential manner.
2. It requires the decision maker to examine all possible outcomes whether desirable or undesirable.
3. It communicates ~~the~~ decision-making process to others in an easy and clear manner indicating each assumption about the future.
4. It displays the logical relationship between the parts of a complex decision and identifies the time sequence in which various actions and subsequent events would occur.
5. It is very useful in situations wherein the initial decision and its outcome affect the subsequent decisions.

Limitations:

1. Decision Tree diagrams become more complicated as the number of decision alternatives increases and more variables are introduced.
2. It becomes highly complicated when interdependent alternatives and dependent variables are present in the problem.
3. There is often inconsistency in assigning probabilities for different events.

Decision making without experimentation

The decision maker and nature can be viewed as the two players of such a game. The alternatives and the possible states of nature can then be viewed as the available strategies for these respective players, where each combination of strategies results in some payoff to player 1. From this viewpoint, the decision analysis framework can be summarised as follows:

1. The decision maker needs to choose one of the decision alternative.
2. Nature then would choose one of the possible states of nature.
3. Each combination of a decision alternative and state of nature would result in pay-off which is given as one of the entries in a pay-off table.
4. This pay-off table should be used to find an optimal alternative for the decision maker.

Game

Game is defined as an activity between 2 or more persons involving activities by each person according to a set of rules at the end of which each person receives some benefit or suffers loss. If in a game, activities are determined by skill, it is called as game of strategy. If the activities are determined by probability, then the game is called as a game of chance.

2 persons: Zero Sum

A game with two persons is called as 2 persons: Zero sum game if the losses of one player is equal to the gains of other player so that the net sum of their gains is zero.

Pay-off matrix

Suppose a player A has m-activities and player B has n-activities, m need not be equal to n. Then, a pay-off matrix is constructed using following conventions:

→ Row designations of the matrix are the activities available to player A.

→ Column designations are the activities available to player B.

→ Cell entry V_{ij} is the payment made by player B to player A when player A selects strategy i and player B selects strategy j.

Ⓑ

Ⓐ	1	2	3	. . . .	n
1	V_{11}	V_{12}	V_{13}	. . . .	V_{1n}
2	V_{21}	V_{22}	V_{23}	. . . .	V_{2n}
3	V_{31}	V_{32}	V_{33}	. . . .	V_{3n}
⋮	⋮	⋮	⋮	⋮	⋮
m	V_{m1}	V_{m2}	V_{m3}	. . . .	V_{mn}

NOTE: In a 2 persons: zero sum game, we always write pay-off matrix wrt player A only as the entries in the pay-off matrix wrt player B will be the negative of the corresponding entries in the pay-off matrix wrt player A.

FUNDAMENTAL THEOREM OF GAME THEORY (Max-Mini Principle)

"EVERY PLAYER TRIES TO MAXIMIZE HIS GAINS AND TRIES TO MINIMIZE THE LOSSES, IF ANY"

Saddle Point

It is the position of such an element in the pay-off matrix which is minimum in its row and maximum in its column.

Optimal Strategies

In a pay-off matrix, if the saddle point is the cell (R, S), then R is the optimal strategy for player A and S is the optimal strategy for player B. The cell entry V_{rs} is called as the value of the game.

Pure Strategy

It is the pre-determined course of action to be employed by the player.

Mixed Strategy

In mixed strategy, the player decides his course of action in accordance with some probability distribution. In mixed strategies the opponent can't be sure of the course of action to be taken on any particular occasion.

Working Rule to find Saddle point

Step 1: Select the minimum element in each row

Step 2: Select the maximum element in each column

Step 3: Find the maximum of minimum values and is denoted by $\underline{V}$.

Step 4: Find the minimum of maximum values and is denoted by $\overline{V}$.

Step 5: If $\underline{V} = \overline{V}$ at a point (r, s), then the point is called as a saddle point.

V_{rs} is called as the value of the game.

If $V_{rs} = 0$, then the game is called as fair game.

SOLVE

-

$$\begin{array}{cc|c|c|c|} & & \multicolumn{3}{c}{\textcircled{B}} \\ & & 1 & 2 & 3 \\ \hline & I & 15 & 2 & 3 \\ \hline \textcircled{A} & II & 6 & 5 & 7 \\ \hline & III & -7 & 4 & 0 \\ \hline \end{array}$$

Solution

$$\begin{array}{cc|ccc|c} & & \multicolumn{3}{c}{\textcircled{B}} & \\ & & 1 & 2 & 3 & \text{Row min} \\ \hline & I & 15 & 2 & 3 & 2 \\ \textcircled{A} & II & 6 & \textcircled{5} & 7 & 5 \\ & III & -7 & 4 & 0 & -7 \\ \hline & \text{Col max} & 15 & 5 & 7 & \end{array}$$

max-min values = $\underline{V}$ = max (2, 5, -7) = 5

min-max values = $\overline{V}$ = min (15, 5, 7) = 5

As $\underline{V} = \overline{V}$, the given pay-off matrix ~~of~~ has saddle point

Optimal strategy for player A = II

Optimal strategy for player B = 2

Value of the game = $\underline{\underline{5}}$

-

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{\textcircled{B}} \\ & & 1 & 2 & 3 & 4 \\ \hline & I & 1 & 2 & 1 & 20 \\ \hline \textcircled{A} & II & 5 & 5 & 4 & 6 \\ \hline & III & 4 & -2 & 0 & -5 \\ \hline \end{array}$$

Solution

$$\begin{array}{cc|cccc|c} & & \multicolumn{4}{c}{\textcircled{B}} & \\ & & 1 & 2 & 3 & 4 & \text{Row min} \\ \hline & I & 1 & 2 & 1 & 20 & 1 \\ \textcircled{A} & II & 5 & 5 & \textcircled{4} & 6 & 4 \\ & III & 4 & -2 & 0 & -5 & -5 \\ \hline & \text{Col max} & 5 & 5 & 4 & 20 & \end{array}$$

max-min values = $\underline{V}$ = max (1, 4, -5) = 4

min-max values = $\overline{V}$ = min (5, 5, 4, 20) = 4

As $\underline{V} = \overline{V}$, game has a saddle point

Optimal strategy for A = II

Optimal strategy for B = 3

Value of the game = $\underline{\underline{4}}$

Ⓑ

		1	2	3	4
	I	1	7	3	4
Ⓐ	II	5	6	4	5
	III	7	2	0	3

Solution

Ⓑ

		1	2	3	4	Row min
	I	1	7	3	4	1
Ⓐ	II	5	6	④	5	4
	III	7	2	0	3	0
	Col max	7	7	4	5	

max-min values = $\underline{V}$ = max (1, 4, 0) = 4

min-max values = $\overline{V}$ = min (7, 7, 4, 5) = 4

As $\underline{V} = \overline{V}$, game has a saddle point

Optimal solution for A = II

optimal strategy for B = 3

Value of the game = $\underline{\underline{4}}$

Ⓑ

		1	2	3
	I	1	2	1
Ⓐ	II	0	-4	-1
	III	1	3	-2

Solution

Ⓑ

$$\begin{array}{c|c|c|c|c}
 & 1 & 2 & 3 & \text{Row min} \\
\hline
I & \textcircled{1} & 2 & \textcircled{1} & 1 \\
Ⓐ\ II & 0 & -4 & -1 & -4 \\
III & 1 & 3 & -2 & -2 \\
\hline
\text{Col max} & 1 & 3 & 1 &
\end{array}$$

max-min values = $\underline{V}$ = max (1, -4, -2) = 1

min-max values = $\overline{V}$ = min (1, 3, 1) = 1

As $\overline{V} = \underline{V}$, game has a saddle point

Optimal strategy for A = I

Optimal strategy for B = 1 or 3

Value of game = $\underline{1}$

- Ⓑ

$$\begin{array}{c|c|c|c}
 & 1 & 2 & 3 \\
\hline
Ⓐ\ I & 6 & 8 & 6 \\
II & 4 & 12 & 2
\end{array}$$

Solution

Ⓑ

$$\begin{array}{c|c|c|c|c}
 & 1 & 2 & 3 & \text{Row min} \\
\hline
Ⓐ\ I & \textcircled{6} & 8 & \textcircled{6} & 6 \\
II & 4 & 12 & 2 & 2 \\
\hline
\text{Col max} & 6 & 12 & 6 &
\end{array}$$

max-min values = $\underline{V}$ = max (6, 2) = 6

min-max values = $\overline{V}$ = min (6, 12, 6) = 6

As $\overline{V} = \underline{V}$, the game has a saddle point

Optimal strategy for A = I

optimal strategy for B = 1 or 3

Value of game = $\underline{6}$

Games without Saddle Points

Method I

Let a given game be a 2×2 game without saddle point with the following pay-off matrix

$$\begin{array}{c|cc} & & Ⓑ \\ & b_1 & b_2 \\ \hline Ⓐ\ a_1 & a & b \\ a_2 & c & d \end{array}$$

Let P_1 and P_2 be the probability that A selects a_1 and a_2 respectively. Let q_1 and q_2 be the probability that B selects b_1 and b_2 respectively.

The above probabilities and the value of the game are calculated using the formulae :

$$p_1 = \frac{d-c}{(a+d)-(b+c)}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} \quad , \quad p_2 = 1-p_1$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} \quad , \quad q_2 = 1-q_1$$

SOLVE

◆

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 8 & -3 \\ A_2 & -3 & 1 \end{array}$$

Solution

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row min} \\ \hline A_1 & 8 & -3 & -3 \\ A_2 & -3 & 1 & -3 \\ \hline \text{Col max} & 8 & 1 & \end{array}$$

max-min value $= \underline{V} = \max(-3,-3) = -3$

min-max value $= \overline{V} = \min(8,1) = 1$

since $\overline{V} \neq \underline{V}$, the game doesnot have saddle point.

$a = 8, \quad b = -3, \quad c = -3, \quad d = 1$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{4}{9-(-6)} = \frac{4}{15} \qquad \therefore p_2 = \frac{11}{15}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{4}{9-(-6)} = \frac{4}{15} \qquad \therefore q_2 = \frac{11}{15}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{8-9}{15} = \frac{-1}{15}$$

optimal strategy for player A = $\left(\frac{4}{15}, \frac{11}{15}\right)$

optimal strategy for player B = $\left(\frac{4}{15}, \frac{11}{15}\right)$

As value of game is −ve, it is advantageous to player B.

- $$\begin{array}{c|c|c|} & B_1 & B_2 \\ \hline A_1 & 5 & 1 \\ \hline A_2 & 3 & 4 \\ \hline \end{array}$$

Solution

$$\begin{array}{c|c|c|c} & B_1 & B_2 & \text{Row min} \\ \hline A_1 & 5 & 1 & 1 \\ \hline A_2 & 3 & 4 & 3 \\ \hline \text{Col max} & 5 & 4 & \end{array}$$

max-min value = $\underline{V}$ = max (1, 3) = 3

min - max value = $\overline{V}$ = min (5, 4) = 4

$\Longrightarrow \overline{V} \neq \underline{V}$

∴ Given game doesn't have a saddle point

$a = 5, \quad b = 1, \quad c = 3, \quad d = 4$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{1}{5} \qquad \therefore p_2 = \frac{4}{5}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{3}{5} // \qquad \therefore q_2 = \frac{2}{5} //$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{17}{5} //$$

Optimal strategy for player A = $\left(\frac{1}{5}, \frac{4}{5}\right)$

Optimal strategy for player B = $\left(\frac{3}{5}, \frac{2}{5}\right)$

As the value of game is +ve, it is advantageous to player A.

•

	B_1	B_2
A_1	2	5
A_2	7	3

Solution

	B_1	B_2	Row min
A_1	2	5	2
A_2	7	3	3
Col max	7	5	

max-min values = $\underline{V}$ = max (2, 3) = 3

min-max values = $\overline{V}$ = min (7, 5) = 5

Since $\overline{V} \neq \underline{V}$, given game doesnot have a saddle point

$a = 2 \quad b = 5 \quad c = 7 \quad d = 3$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-4}{(5)-12} = \frac{4}{7} // \qquad \therefore p_2 = \frac{3}{7}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-2}{-7} = \frac{2}{7} // \qquad \therefore q_2 = \frac{5}{7} //$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{6-35}{-7} = \frac{29}{7} //$$

Optimal strategy for A = $\left(\frac{4}{7}, \frac{3}{7}\right)$

Optimal strategy for B = $\left(\frac{2}{7}, \frac{5}{7}\right)$

∴ Value of game is +ve, it is advantageous to A.

	B_1	B_2
A_1	-4	6
A_2	2	-3

Solution

	B_1	B_2	Row min
A_1	-4	6	-4
A_2	2	-3	-3
Col max	2	6	

max-min value = $\underline{V} = \max(-4, -3) = -3$

min-max value = $\overline{V} = \min(2, 6) = 2$

Since $\overline{V} \neq \underline{V}$, given game doesnot have a saddle point.

$a = -4 \quad b = 6 \quad c = 2 \quad d = -3$

$$p_1 = \frac{d - c}{(a+d) - (b+c)} = \frac{-3-2}{-7-8} = \frac{1}{3} \qquad \therefore p_2 = \frac{2}{3}$$

$$q_1 = \frac{d - b}{(a+d) - (b+c)} = \frac{-3-6}{-15} = \frac{3}{5} \qquad \therefore q_2 = \frac{2}{5}$$

$$V = \frac{ad - bc}{(a+d) - (b+c)} = \frac{12 - 12}{-15} = 0$$

Optimal Strategy for A = $\left(\frac{1}{3}, \frac{2}{3}\right)$

Optimal strategy for B = $\left(\frac{3}{5}, \frac{2}{5}\right)$

Value of the game is 0 ⇒ It is a fair game

-

	B_1	B_2
A_1	6	-3
A_2	-3	0

Solution:

	B_1	B_2	Row min
A_1	6	-3	-3
A_2	-3	0	-3
Col max	6	0	

max-min value $= \underline{V} = \max(-3, -3) = -3$

min-max value $= \bar{V} = \min(6, 0) = 0$

Since $\bar{V} \neq \underline{V}$, the given game doesnot have a saddle point

$a = 6 \quad b = -3 \quad c = -3 \quad d = 0$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{0-(-3)}{6-(-6)} = \frac{3}{12} = \frac{1}{4} \qquad \therefore p_2 = \frac{3}{4}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{0-(-3)}{12} = \frac{1}{4} \qquad \therefore q_2 = \frac{3}{4}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{0-9}{12} = -\frac{3}{4}$$

Optimal strategy for A $= \left(\frac{1}{4}, \frac{3}{4}\right)$

Optimal strategy for B $= \left(\frac{1}{4}, \frac{3}{4}\right)$

Since value of the game is -ve, it is advantageous to B

-

	B_1	B_2
A_1	2	5
A_2	4	1

Solution

	B_1	B_2	Row min
A_1	2	5	2
A_2	4	1	1

max-min values = $\underline{V}$ = max (2, 1) = 1

min-max values = $\bar{V}$ = min (4, 5) = 4

Since $\bar{V} \neq \underline{V}$, the given game doesnot have a saddle point. $a = 2$ $b = 5$ $c = 4$ $d = 1$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-3}{3-9} = \frac{1}{2} \qquad \therefore p_2 = \frac{1}{2}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-4}{-6} = \frac{2}{3} \qquad \therefore q_2 = \frac{1}{3}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{2-20}{-6} = 3$$

Optimal Strategy for A is $\left(\frac{1}{2}, \frac{1}{2}\right)$

optimal strategy for B is $\left(\frac{2}{3}, \frac{1}{3}\right)$

Since value of the game is positive, it is advantageous to A.

- Two players A and B match coins. If coins match, then player A gets 1 rupee and if coins doesn't match, player B gets 1 rupee. Determine the optimum strategies for both the players and also find the value of the game.

Solution

		Ⓑ		
		H	T	Row min
Ⓐ	H	1	-1	-1
	T	-1	1	-1
col max		1	1	

max-min values = $\underline{V}$ = max (−1, −1) = −1

min-max values = $\bar{V}$ = min (1, 1) = 1

Since $\bar{V} \neq \underline{V}$, the given game doesnot have a saddle point.

$a = 1 \quad b = -1 \quad c = -1 \quad d = 1$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2}{2-(-2)} = \frac{1}{2} \qquad \therefore p_2 = \frac{1}{2}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{1}{2} \qquad \therefore q_2 = \frac{1}{2}$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{1-1}{4} = 0$$

Optimal strategy for A = $\left(\frac{1}{2}, \frac{1}{2}\right)$

optimal strategy for B = $\left(\frac{1}{2}, \frac{1}{2}\right)$

Since the value of game = 0, it is a fair game.

DOMINANCE RULE TO SOLVE $m \times n$ GAMES

We use the following rules to reduce a given game to a 2×2 game or a 1×1 game

Rule 1: If all the elements of the i^{th} row are $\leq$ the corresponding elements of the j^{th} row, we say that j^{th} strategy dominates i^{th} strategy and hence we delete i^{th} row

$$\boxed{\text{If } R_i \leq R_j, \text{ delete } R_i}$$

Rule 2: If all the elements of the p^{th} column are $\geq$ corresponding elements of the q^{th} column, then we say that q^{th} strategy dominates p^{th} strategy and hence we delete p^{th} strategy.

$$\boxed{\text{If } C_p \geq C_q, \text{ delete } C_p}$$

Rule 3: If row dominance and column dominance cannot reduce a game, we take averages

(a) If all the elements of a row R_i are $\leq$ the average of two or more rows, then we say that the group of rows dominate the i^{th} row and hence we delete i^{th} row.

(b) If all the elements of a column C_i are $\geq$ the averages of two or more columns, then we say that the group of columns dominate the i^{th} column and hence we delete i^{th} column.

SOLVE BY DOMINANCE RULE

	b_1	b_2	b_3
a_1	+5	20	-10
a_2	10	6	2
a_3	20	15	18

Solution

	~~c_1~~	c_2	c_3
R_1	~~5~~	20	-10
~~R_2~~	~~10~~	~~6~~	~~2~~
R_3	~~20~~	15	18

We note $R_2 \le R_3$ $\therefore$ Delete R_2

We note $C_3 \le C_1$ $\therefore$ Delete C_1

Further reduction of the game by dominance rule is not possible. Hence the given game is reduced to the following 2x2 game.

	b_2	b_3
a_1	20	-10
a_3	15	18

$a = 20$ $b = -10$ $c = 15$ $d = 18$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{3}{38-5} = \frac{3}{33} = \frac{1}{11} // \qquad \therefore p_2 = \frac{10}{11} //$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{28}{33} // \qquad \therefore q = \frac{5}{33} //$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{360+150}{33} = \frac{510}{33} = \frac{170}{11} //$$

Optimal strategy for A = $\left(\frac{1}{11}, 0, \frac{10}{11}\right)$

Optimal strategy for B = $\left(0, \frac{28}{33}, \frac{5}{33}\right)$

Since value of the game is +ve, it is advantageous to A.

	b_1	b_2	b_3	b_4
a_1	2	-2	4	1
a_2	6	1	12	3
a_3	-3	2	0	6
a_4	2	-3	7	1

Solution:

	b_1	b_2	b_3	b_4
~~a_1~~	~~2~~	~~-2~~	~~4~~	~~1~~
a_2	6	1	~~12~~	~~3~~
a_3	-3	2	~~0~~	~~6~~
~~a_4~~	~~2~~	~~-3~~	~~7~~	~~1~~

we note $a_1 \leq a_2$ $\therefore$ delete a_1

$a_4 \leq a_2$ $\therefore$ delete a_4

$b_3 \geq b_1$ $\therefore$ delete b_3

$b_4 \geq b_2$ $\therefore$ delete b_4

Further reduction of the game by dominance rule is not possible. Hence the given game is reduced to the following 2×2 game.

	b_1	b_2
a_2	6	1
a_3	-3	2

$a = 6 \quad b = 1 \quad c = -3 \quad d = 2$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2+3}{8-(-2)} = \frac{5}{10} = \frac{1}{2} \qquad \therefore p_2 = \frac{1}{2}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{1}{10} \qquad \therefore q_2 = \frac{9}{10}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{12+3}{10} = \frac{15}{10} = \frac{3}{2}$$

Optimal Strategy for A $= \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ //

Optimal strategy for B $= \left(\frac{1}{10}, \frac{9}{10}, 0, 0\right)$ //

Since the value of the game is +ve, it is advantageous to player A.

•

	b_1	b_2	b_3	b_4	b_5
a_1	2	4	3	8	4
a_2	5	6	3	7	8
a_3	6	7	9	8	7
a_4	4	2	8	4	3

Solution:

	b_1	~~b_2~~	~~b_3~~	~~b_4~~	~~b_5~~
~~a_1~~	~~2~~	~~4~~	~~3~~	~~8~~	~~4~~
~~a_2~~	~~5~~	~~6~~	~~3~~	~~7~~	~~8~~
a_3	6	~~7~~	~~9~~	~~8~~	~~7~~
~~a_4~~	~~4~~	~~2~~	~~8~~	~~4~~	~~3~~

We note that $a_1 \leq a_3$ $\therefore$ delete a_1

$a_4 \leq a_3$ $\therefore$ delete a_4

$b_2 \geq b_1$ $\therefore$ delete b_2

$b_4 \geq b_1$ $\therefore$ delete b_4

$b_5 \geq b_1$ $\therefore$ delete b_5

$a_2 \leq a_3$ $\therefore$ delete a_2

$b_3 \geq b_1$ $\therefore$ delete b_3

Further reduction of the game by dominance rule is not possible. Hence the given game is reduced to the following 1×1 game

	b_1
a_3	6

Optimal strategy for A = a_3
optimal strategy for B = b_1
$\therefore$ value of the game = 6

Since the value of the game is positive, it is advantageous to player A.

•

	b_1	b_2	b_3
a_1	1	7	2
a_2	6	2	7
a_3	5	2	6

Solution

	b_1	b_2	~~b_3~~
a_1	1	7	~~2~~
a_2	6	2	~~7~~
~~a_3~~	~~5~~	~~2~~	~~6~~

We note that $b_3 \geqslant b_1$ $\therefore$ delete b_3
$a_3 \leq a_2$. $\therefore$ delete a_3

Further reduction of the game by dominance rule is not possible. Hence the given game is reduced ~~to~~ following 2×2 game.

	b_1	b_2
a_1	1	7
a_2	6	2

$a = 1$ $\quad b = 7$ $\quad c = 6$ $\quad d = 2$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-4}{3-13} = \frac{-4}{-10} = \frac{2}{5} // \qquad \therefore p_2 = \frac{3}{5} //$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-5}{-10} = \frac{1}{2} // \qquad \therefore q_2 = \frac{1}{2} //$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{2-42}{-10} = \frac{-40}{-10} = 4 //$$

Optimal strategy for A = $\left(\frac{2}{5}, \frac{3}{5}, 0\right)$

Optimal strategy for B = $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Since the value of game is +ve, it is advantageous to A.

-
$$\begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline a_1 & 2 & 8 & 3 \\ a_2 & 6 & 2 & 8 \\ a_3 & 4 & 1 & 6 \end{array}$$

Solution

$$\begin{array}{c|ccc} & b_1 & b_2 & \cancel{b_3} \\ \hline a_1 & 2 & 8 & \cancel{3} \\ a_2 & 6 & 2 & \cancel{8} \\ \cancel{a_3} & \cancel{4} & \cancel{1} & \cancel{6} \end{array}$$

We note $a_3 \leq a_2$. ∴ delete a_3

$b_3 \geq b_1$ ∴ delete b_3

Further reduction of the game by dominance rule is not possible. Hence the given game is reduced to the following 2×2 game.

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & 2 & 8 \\ a_2 & 6 & 2 \end{array}$$

$a = 2 \quad b = 8 \quad c = 6 \quad d = 2$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-4}{4-14} = \frac{-4}{-10} = \frac{2}{5} \quad ∴ p_2 = \frac{3}{5}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-6}{-10} = \frac{3}{5} \quad ∴ q_2 = \frac{2}{5}$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{4-48}{-10} = \frac{-44}{-10} = \frac{22}{5}$$

optimal strategy for A = $\left(\frac{2}{5}, \frac{3}{5}\right)$

optimal strategy for B = $\left(\frac{3}{5}, \frac{2}{5}\right)$

Since the value of the game is positive, it is advantageous to A

- Determine the optimal strategy for each player by successively eliminating the dominated strategy. Indicate the order in which the strategies are eliminated. [JAN 2010]

	b_1	b_2	b_3
a_1	1	2	0
a_2	2	-3	-2
a_3	0	3	-1

Solution:

	~~c_1~~	~~c_2~~	c_3
R_1	~~1~~	~~2~~	0
~~R_2~~	~~2~~	~~-3~~	~~-2~~
~~R_3~~	~~0~~	~~3~~	~~-1~~

We note that (i) $C_1 \geq C_3$. ∴ delete c_1

$\Rightarrow$ strategy B_1 is eliminated

(ii) $R_2 \leq R_3$. ∴ delete R_2

$\Rightarrow$ strategy A_2 is eliminated

(iii) $C_2 \geq C_3$. ∴ delete C_2

$\Rightarrow$ strategy B_2 is eliminated

(iv) $R_3 \leq R_1$ ∴ delete R_3

$\Rightarrow$ strategy A_3 is eliminated

optimal strategy for A is a_1 //

optimal strategy for B is b_3 //

Value of the game = 0 //

∴ It is a fair game

	b_1	b_2	b_3	b_4	b_5
a_1	1	3	2	7	4
a_2	3	4	1	5	6
a_3	6	5	7	6	5
a_4	2	0	6	3	1

Solution:

	b_1	b_2	b_3	b_4	b_5
a_1	1	3	2	7	4
a_2	3	4	1	5	6
a_3	6	5	7	6	5
a_4	2	0	6	3	1

We note $b_4 \geqslant b_1$ $\therefore$ delete b_4

$b_5 \geqslant b_2$ $\therefore$ delete b_5

$a_1 \leq a_3$ $\therefore$ delete a_1

$a_2 \leq a_3$ $\therefore$ delete a_2

$a_4 \leq a_3$ $\therefore$ delete a_4

$b_3 \geqslant b_2$ $\therefore$ delete b_3

$b_1 \geqslant b_2$ $\therefore$ delete b_1

Further reduction of the game by dominance rule is not possible. Now it is reduced to a 1×1 game.

$$\begin{array}{cc} & b_1 \\ a_3 & \boxed{5} \end{array}$$

Optimal strategy for A is a_3

Optimal strategy for B is b_2

Value of game = 5

Since the value of the game is +ve, it is advantageous to player A.

- $$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 6 & 4 & 8 & 0 \\ a_2 & 6 & 8 & 4 & 8 \\ a_3 & 8 & 4 & 8 & 0 \\ a_4 & 0 & 8 & 0 & 16 \end{array}$$

Solution

$$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline \cancel{a_1} & \cancel{6} & \cancel{4} & \cancel{8} & \cancel{0} \\ \cancel{a_2} & \cancel{6} & \cancel{8} & \cancel{4} & \cancel{8} \\ a_3 & \cancel{8} & \cancel{4} & 8 & 0 \\ a_4 & \cancel{0} & \cancel{8} & 0 & 16 \end{array}$$

We note that $R_1 \leq R_3$. ∴ delete R_1

$C_1 \geq C_3$ ∴ delete C_1

Further reduction of the game is not possible.

∴ We shall take the averages

$$C_2 \geq \left(\frac{C_3 + C_4}{2}\right) \quad ∴ \text{delete } C_2$$

$$R_2 \leq \left(\frac{R_3 + R_4}{2}\right) \quad ∴ \text{delete } R_2$$

Further reduction of the game is not possible by dominance rule.

∴ It is reduced to 2×2 game.

$$\begin{array}{c|cc} & b_3 & b_4 \\ \hline a_3 & 8 & 0 \\ a_4 & 0 & 16 \end{array}$$

$a = 8 \quad b = 0 \quad c = 0 \quad d = 16$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{16}{24} = \frac{2}{3} \qquad ∴ p_2 = \frac{1}{3}$$

$$q_1 = \frac{d-b}{(a+d)\cancel{+}-(b+c)} = \frac{16}{24} = \frac{2}{3} \qquad ∴ q_2 = \frac{1}{3}$$

$$V = \frac{ad - bc}{(a+d)-(b+c)} = \frac{128}{24} = \frac{16}{3} //$$

Optimal strategy for A $= \left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$ //

optimal strategy for B $= \left(0, 0, \frac{2}{3}, \frac{1}{3}\right)$ //

Value of the game $= \frac{16}{3}$ //

Since it is +ve, it is advantageous to player A.

ASSIGNMENT

-

	C_1	C_2	C_3	C_4
R_1	3	2	4	0
R_2	3	4	2	4
R_3	4	2	4	0
R_4	0	4	0	8

-

	C_1	C_2	C_3	C_4	C_5	C_6
R_1	4	2	0	2	1	1
R_2	4	3	1	3	2	2
R_3	4	3	7	-5	1	2
R_4	4	3	4	-1	2	2
R_5	4	3	3	-2	2	2

-

	b_1	b_2	b_3	b_4
a_1	36	65	25	5
a_2	30	20	15	0
a_3	40	50	0	10
a_4	55	60	10	15

USN

06CS661

Sixth Semester B.E. Degree Examination, Dec.09/Jan.10

Operations Research

Time: 3 hrs. Max. Marks:100

Note: 1. *Answer any FIVE full questions, selecting at least TWO questions from each part.*
2. *Any missing data may be assumed suitably.*

Part - A

1 a. What is operations research? Mention six phases of an operations research study. **(05 Marks)**

b. Formulate a linear programming model for the problem given below. The Apex television company has to decide on the number of 27-inch and 20-inch sets to be produced at one of its factories. Market research indicates that at most 40 of the 27-inch sets and 10 of 20-inch sets can be sold per month. The maximum number of work hours available is 500 per month. A 27-inch set requires 20 work hours and 20-inch set requires 10 work hours. Each 27-inch set sold produces a profit of \$120 and each 20-inch produces a profit of \$80. A wholesaler agreed to purchase all the television sets produced if the numbers do not exceed the maxima indicated by market research. **(05 Marks)**

c. Use graphical method to solve the following LPP:

Maximize $z = 3x_1 + 5x_2$

Subject to $x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1 \ge 0,\ x_2 \ge 0$ **(05 Marks)**

d. Write the meaning of following terms with respect to a LPP. Give example for each:
i) Feasible solution ii) Infeasible solution. iii) Feasible region.
iv) Optimal solution. v) CPF solution. **(05 Marks)**

2 a. Write four assumptions of linear programming. **(04 Marks)**

b. Write six key solution concepts of simplex method. **(06 Marks)**

c. Solve the following LPP using simplex method in tabular form:

Maximize $z = 5x_1 + 4x_2$

Subject to $6x_1 + 4x_2 \le 24$

$x_1 + 2x_2 \le 6$

$-x_1 + x_2 \le 1$

$x_2 \le 2$ and $x_1 \ge 0,\ x_2 \ge 0$ **(10 Marks)**

3 a. Using Big M method solve the following:

Minimize $z = 3x_1 + 2x_2 + x_3$

Subject to $x_1 + x_2 = 7$

$3x_1 + x_2 + x_3 \ge 10$

and $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$ **(12 Marks)**

b. Explain the typical steps in post optimality analysis for linear programming studies. **(08 Marks)**

4 a. Apply revised simplex method to solve the following problem:

Maximize $z = 4x_1 + 3x_2 + 6x_3$

Subject to $3x_1 + x_2 + 3x_3 \le 30$

$2x_1 + 2x_2 + 3x_3 \le 40$

and $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$ **(12 Marks)**

b. Explain key relationships between primal and dual problems. **(08 Marks)**

2. Any revealing of identification, appeal to evaluator and/or equations written eg, 42+8=50, will be treated as malpractice.

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Part - B

5 a. Write a procedure for sensitivity analysis. **(08 Marks)**

b. Use dual simplex method to solve the following:

Maximize $z = -4y_1 - 12y_2 - 18y_3$

Subject to $y_1 + 3y_3 \geq 3$

$2y_2 + 2y_3 \geq 5$

and $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$ **(12 Marks)**

6 a. Suppose that England, France and Spain produce all the wheat; barley and oats in world. The world demand for wheat requires 125 million acres of land devoted to wheat production; similarly, 60 million acres of land are required for barley and 75 million acres of land for oats. The total amounts of land available for these purposes in England, France and Spain are 70 million acres, 110 million acres, 80 million acres respectively. The number of hours of labor needed in England, France and Spain to produce an acre of wheat is 18, 13 and 16 respectively. The number of hours of labor needed in England, France and Spain to produce an acre of barley is 15, 12 and 12 respectively. The number of hours of labor needed in England, France and Spain to produce an acre of oats is 12, 10 and 16 respectively. The labor cost per hour in producing wheat is \$9.00, \$7.20 and \$9.90 in England, France and Spain respectively. The labor cost per hour in producing barley is \$8.10, \$9.00 and \$8.40 in England, France and Spain respectively. The labor cost per hour in producing oats is \$6.90, \$7.50 and \$6.30 in England, France and Spain respectively. The problem is to allocate land use in each country so as to meet the world food requirement and minimize the total labor cost.

i) Formulate this problem as a transportation problem by constructing the appropriate parameter table.

ii) Starting with the north west corner rule, interactively apply the transportation simplex method to obtain an optimal solution. **(12 Marks)**

b. Write different steps in Hungarian algorithm to solve an assignment problem. **(08 Marks)**

7 a. Explain basic characteristics of two person, zero sum game. For the game having following pay off table, determine the optimal strategy for each player by successively eliminating dominated strategies. Indicate the order in which you eliminate strategies. **(10 Marks)**

Player – 2

Player - 1

	1	2	3
1	1	2	0
2	2	-3	-2
3	0	3	-1

b. Explain how to construct a decision tree and how it is used for decision analysis. **(10 Marks)**

8 Explain briefly:

a. Metaheuristics, its nature, advantage and disadvantage.

b. Tabu search algorithm.

c. Simulated annealing algorithm.

d. Genetic algorithm. **(20 Marks)**

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06CS/IS661

USN

Sixth Semester B.E. Degree Examination, May/June 2010
Operations Research

Time: 3 hrs. Max. Marks:100

Note: ***Answer any FIVE full questions, selecting at least TWO questions from each part.***

PART – A

1 a. Explain the following :
   i) Origin, nature and impact of OR.
   ii) Defining the problem and gathering the data. **(10 Marks)**

b. A farmer has to plant two kinds of trees P and Q in a land of 4000 sq.m. area. Each P tree requires at least 25 sq.m and Q tree requires at least 40 sq.m. of land. The annual water requirements of P tree is 30 units and of Q tree is 15 units per tree, while at most 3000 units of water is available. It is also estimated that the ratio of the number Q trees to the number of P trees should not be less than $\frac{6}{19}$ and should not be more than $\frac{17}{8}$. The return per tree from P is expected to be one and half times as mush as from Q tree.
Formulate the problem as a LP model. **(10 Marks)**

2 a. Solve the following LPP by simplex method :
Maximize $Z = 3x_1 + 2x_2$
Subject to $x_1 + x_2 \leq 4$,
$x_1 - x_2 \leq 2$
$x_1, x_2 \geq 0$. **(10 Marks)**

b. Solve the following LPP by simplex method :
Maximize $Z = 6x_1 + 8x_2$
Subject to $2x_1 + 8x_2 \leq 16$
$2x_1 + 4x_2 \leq 8$
$x_1, x_2 \geq 0$ **(10 Marks)**

3 a. Explain in detail the computer implementation of simplex method and available software option for linear programming. **(10 Marks)**

b. Explain the postoptimality analysis of linear programming. **(05 Marks)**

c. Explain the two phase technique procedure of solve LPP in simplex method. **(05 Marks)**

4 a. Explain the relation between the solution of the primal and the dual. **(05 Marks)**

b. Find the dual of the following problem :
Minimize $Z = 2x_2 + 5x_3$
Subject to $x_1 + x_2 \geq 2$
$2x_1 + x_2 + 6x_3 \leq 6$
$x_1 - x_2 + 3x_3 = 4$
$x_1, x_2, x_3 \geq 0$ **(05 Marks)**

c. Explain the computational procedure of revised simplex method in standard form. **(10 Marks)**

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PART – B

5 a. Use dual simplex method and solve the following LPP :
Minimize $Z = 3x_1 + x_2$
Subject to $x_1 + x_2 \geq 1$
$2x_1 + 3x_2 \geq 2$
$x_1, x_2 \geq 0$ **(10 Marks)**

b. Explain the role of duality theory in sensitivity analysis. **(05 Marks)**

c. Explain how sensitivity analysis has been applied. **(05 Marks)**

6 a. Find an initial solution to the following transportation problem :

Destination

		D_1	D_2	D_3	D_4	D_5		
Origin	O_1	7	6	4	5	9	40	Supply
	O_2	8	5	6	7	8	30	
	O_3	6	8	9	6	5	20	
	O_4	5	2	7	8	6	10	
		30	30	15	20	5		

Demand

(10 Marks)

b. The owner of a small machine shop has four machines available to assign for the jobs. Five jobs are offered to assign, with the expected profits in hundreds of rupees for each machine on each job being as follows :

Job

		1	2	3	4	5
Machines	A	6.2	7.8	*	10.1	8.2
	B	7.0	8.4	6.5	7.5	6.0
	C	8.7	9.2	11.1	7.0	8.2
	D	*	6.4	8.7	7.7	8.0

* indicates that machine A and D cannot perform the jobs 3 and 1 respectively. Find the assignment of jobs to machinies that will result in the maximum profit. **(10 Marks)**

7 a. Solve the following game graphically : **(10 Marks)**

Player B

		B_1	B_2	B_3
Player A	A_1	2	6	22
	A_2	16	10	4

b. Explain in detail decision making without experimentation. **(05 Marks)**

c. Explain the details of solving simple games in game theory. **(05 Marks)**

8 a. Explain in detail, the minimum spanning tree problem with constraints. **(06 Marks)**

b. Outline the general procedure for generating a child from a pair of parents. **(06 Marks)**

c. Explain the number of details that need to be worked out to fit structure of the problem. **(08 Marks)**

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06CS661

USN ☐☐☐☐☐☐☐☐☐☐

Sixth Semester B.E. Degree Examination, December 2010

Operations Research

Time: 3 hrs. Max. Marks:100

Note: *1. Answer any FIVE full questions, selecting at least TWO questions from each part.*
2. Missing data, if any, may be suitably assumed.

PART – A

1 a. What is operations research? Explain the six phases of a study. **(07 Marks)**

b. Use the graphical method to solve the problem :

Maximise $Z = 10x_1 + 20x_2$

Subject to $-x_1 + 2x_2 \le 15$

$x_1 + x_2 \le 12$

$5x_1 + 3x_2 \le 45$

and $x_1, x_2 \ge 0$. **(07 Marks)**

c. Explain the linear programming model. **(06 Marks)**

2 a. Explain the steps needed to find feasible solution using simplex method. **(06 Marks)**

b. Work through the simplex method step by step to solve the following problem :

Minimize $Z = x_1 - 3x_2 + 3x_3$

Subject to $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 + 4x_2 \ge -12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

and $x_1, x_2, x_3 \ge 0$. **(14 Marks)**

3 a. Solve, by using Big – M method, the following linear programming problem :

Maximise $Z = -2x_1 - x_2$

Subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \le 4$

and $x_1, x_2 \ge 0$. **(07 Marks)**

b. Use two-phase method to solve the problem :

Minimize $Z = 0.4x_1 + 0.5x_2$

Subject to $0.3x_1 + 0.1x_2 \le 2.7$

$0.5x_1 + 0.5x_2 = 6$

$0.6x_1 + 0.4x_2 \ge 6$

and $x_1, x_2 \ge 0$. **(13 Marks)**

4 a. Apply revised simplex method to solve the following problem :

Maximise $Z = 6x_1 - 2x_2 + 3x_3$

Subject to $2x_1 - x_2 + 2x_3 \le 2$

$x_1 + 4x_3 \le 4$

and $x_1, x_2, x_3 \ge 0$. **(14 Marks)**

b. Explain :

i) Weak duality property

ii) Strong duality property

iii) Complementary solutions property. **(06 Marks)**

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

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PART – B

5 a. Explain the key relationships between primal and dual problems. **(06 Marks)**

b. Solve the following problem by dual simplex method.

Minimise $Z = 2x_1 + x_2$

Subject to $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

and $x_1, x_2 \geq 0.$

(14 Marks)

6 a. Write different steps in Hungarian algorithm to solve an assignment problem. **(08 Marks)**

b. Find the initial basic feasible solution of transportation problem where cost – matrix is given below :

		Destination				
		A	B	C	D	Supply
Origin	I	1	5	3	3	34
	II	3	3	1	2	15
	III	0	2	2	3	12
	IV	2	7	2	4	19
	Demand	21	25	17	17	

(12 Marks)

7 a. Explain the various variations in solving games, with examples. **(08 Marks)**

b. Solve the game whose payoff matrix to the player A is given below :

		B		
		I	II	III
A	I	1	7	2
	II	6	2	7
	III	5	2	6

(12 Marks)

8 Explain briefly :

a. Decision trees

b. Tabu search algorithm

c. Genetic algorithm

d. Metaheuristics.

(20 Marks)

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USN ☐☐☐☐☐☐☐☐☐☐

06CS/IS661

Sixth Semester B.E. Degree Examination, June/July 2011

Operations Research

Time: 3 hrs.

Max. Marks:100

Note: *Answer FIVE full questions, selecting at least TWO questions from each part.*

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

PART – A

1 a. What are the different phases of OR? Explain them briefly. **(08 Marks)**

b. Define the following with reference to LPP
i) Unbounded solution. ii) Feasible solution. iii) Slack Variable. **(04 Marks)**

c. ABC firm manufactures three products P_1, P_2 and P_3. The profits are Rs. 30, Rs. 20 and Rs. 40 respectively. The firm has two machines M1 and M2 and requires processing time in minutes for each machine on each product and total machine available minutes on each machine are given below.

Machine	Machine minutes required			Total machine minutes available
	P1	P2	P3	
M1	4	3	5	2000
M2	2	2	4	2500

The firm must manufacture at least 100 P_1's and 200 P_2's and 50 P_3's but not more than 150 P_1's. Setup LP model to solve by simplex method. **(08 Marks)**

2 a. Briefly explain assumptions required in Linear programming models. **(05 Marks)**

b. Use graphical method to solve the following:

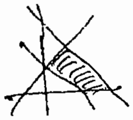
Maximize $z = x_1 + \frac{x_2}{2}$

subject to $3x_1 + 2x_2 \le 12$

$5x_1 \le 10, \quad x_1 + x_2 \le 18$

$-x_1 + x_2 \ge 4, \quad x_1 \text{ and } x_2 \ge 0$

12.5.



(12 Marks)

c. Why is simplex method a better technique than graphical for most real case? Explain **(03 Marks)**

3 a. Explain the concept of degeneracy in simplex method. **(04 Marks)**

b. Use penalty method to solve the following LPP

Minimize $z = 5x_1 + 3x_2$

Subject to $2x_1 + 4x_2 \le 12$

$2x_1 + 2x_2 = 10, \quad 5x_1 + 2x_2 \ge 10$

$x_1 \text{ and } x_2 \ge 0$

45
5
0

(16 Marks)

4 a. Construct the dual problem for the following LPP

Maximize $Z = 16x_1 + 14x_2 + 36x_3 + 6x_4$

Subject to $14x_1 + 4x_2 + 14x_3 + 8x_4 = 21$; $13x_1 + 17x_2 + 80x_3 + 2x_4 \le 48$

$x_1, x_2 \ge 0$; x_3 ; x_4 unrestricted. **(06 Marks)**

b. Use revised simplex method to solve the following LPP

Maximize $z = x_1 + 2x_2$

subject to $x_1 + x_2 \le 3, \quad x_1 + 2x_2 \le 5$

$3x_1 + x_2 \le 6, \quad x_1, x_2 \ge 0$ **(14 Marks)**

68(72)

PART – B

5 a. Briefly discuss about sensitivity analysis. **(06 Marks)**

b. Find the maximum of $z = 6x_1 + 8x_2$

subject to $5x_1 + 2x_2 \leq 20$

$x_1 + 2x_2 \leq 10$

$x_1 \,\&\, x_2 \geq 0$

by solving its dual problem using simplex method. **(14 Marks)**

6 a. Define feasible solution, basic feasible solution, non-degenerate solution and optimal solution in a Transportation problem. **(06 Marks)**

b. A product is produced by 4 factories F₁, F₂, F₃ and F₄. Their unit production costs are Rs. 2, 3, 1 and 5 respectively. Production capacity of the factories are 50, 70, 30 and 50 units respectively. The product is supplied to 4 stores S₁, S₂, S₃ and S₄, the requirements of which are 25, 35, 105 and 20 respectively. Unit costs of transportation are given below.

Stores / Factories	S_1	S_2	S_3	S_4
F_1	2	4	6	11
F_2	10	8	7	5
F_3	13	3	9	12
F_4	4	6	8	3

965

786

Find the transportation plan such that the total production and transportation cost is minimum. **(14 Marks)**

7 a. Solve the following assignment problem. If it is treated as a salesman problem and the cell entries represent cost in rupees, find the least cost route such that salesman does not visit any city twice.

	A	B	C	D	E
A	–	2	5	7	1
B	6	–	3	8	2
C	8	7	–	4	7
D	12	4	6	–	5
E	1	3	2	8	–

(14 Marks)

b. Explain the following

i) Minimax and Maximin principles.

ii) Pure and Mixed strategies.

iii) Two persons zero sum game. **(06 Marks)**

8 a. Write a brief note on Tabu search algorithm. **(04 Marks)**

b. Reduce the following (2 × n) game to (2 × 2) game by graphical method and hence solve.

		B				
		I	II	III	IV	V
A	I	2	-1	5	-2	6
	II	-2	4	-3	1	0

(08 Marks)

c. A news paper boy has the following probabilities of selling a magazine

No. of copies sold	10	11	12	13	14
Probability	0.10	0.15	0.20	0.25	0.30

Cost of a copy is 30 paise and sale price is 50 paise. He can not return unsold copies. How many copies should he order? **(08 Marks)**

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69 (72)

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06CS/IS661

Sixth Semester B.E. Degree Examination, December 2011

Operations Research

Time: 3 hrs. Max. Marks:100

Note: ***Answer any FIVE full questions, selecting at least TWO questions from each part.***

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator and /or equations written eg, 42+8 = 50, will be treated as malpractice.

PART – A

1 a. What is operations research? Explain the impact of OR. **(06 Marks)**

b. A farmer has 100 acre farm. He can sell all tomatoes, lettuce, or radishes he can raise. The price he can obtain is ₹1.00 per kg for tomatoes, ₹0.75 a head for lettuce and ₹2.00 per kg for radishes. The average yield per acre is 2000kg of tomatoes, 3000 heads of lettuce and 1000kg of radishes. Fertilizer is available at ₹0.50 per kg and the amount required per acre is 100kg each for tomatoes and lettuce and 50kg for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days fort lettuce. A total of 400 man-days of labour are available at ₹20.0 per man-day. Formulate this problem as a linear programming model to maximize the farmer's total profit. **(06 Marks)**

c. Old hens can be bought at ₹2 each and young ones at ₹5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs ₹1 per week to feed. You have only ₹80 to spend for buying hens. How many of each kind should you buy to give a profit of more than ₹6 per week assuming that you cannot house more than 20 hens. Formulate the problem as an LPP and solve graphically. **(08 Marks)**

2 a. TOYCO assembles three types of toys – trains, trucks and cars, using three operations. The daily limits on the available times for the three operations are 430, 460 and 420 minutes respectively, and the revenues per unit of toy train, truck and car are \$3, \$2 and \$5 respectively. The assembly times per train at the three operations are 1, 3 and 1 minutes respectively. The corresponding times per truck and per car are (2, 0, 4) and (1, 2, 0) minutes (a zero time indicate that the operation is not used). Formulate the problem as LPP and solve using the simplex method. **(10 Marks)**

b. Explain the special cases that arise in the use of simplex method. **(10 Marks)**

3 a. Solve the problem, using the Big-M method.
Maximize $Z = 6x_1 + 4x_2$
Subject to constraints, $2x_1 + 3x_2 \le 30$; $3x_1 + 2x_2 \le 24$; $x_1 + x_2 \ge 3$; $x_1 \ge 0$; $x_2 \ge 0$
Find at least two solutions. **(10 Marks)**

b. Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the minimum cost of the product mix. Formulate the problem and solve using the two phase method. **(10 Marks)**

4 a. Use the revised simplex method to solve the following LPP:
Maximize $Z = 6x_1 - 2x_2 - 3x_3$
Subject to constraints, $2x_1 - x_2 + 2x_3 \le 2$; $x_1 + 4x_3 \le 4$; $x_1 , x_2 , x_3 \ge 0$. **(10 Marks)**

b. Obtain the dual solution directly, using the inverse from solution of the primal.
Maximize $Z = 5x_1 + 2x_2 + 3x_3$
Subject to constraints, $x_1 + 5x_2 + 2x_3 = 30$; $x_1 - 5x_2 - 6x_3 \le 40$; $x_1 , x_2 , x_3 \ge 0$. **(10 Marks)**

PART – B

5 a. Explain the parametric analysis with respect to change in c_j and b_j parameters. **(08 Marks)**

b. Obtain the optimal solution, using the dual simplex method for the dual problem of the following:
Maximize $Z = 3x_1 + 5x_2$
Subject to constraints, $x_1 \le 4$; $2x_2 \le 12$; $3x_1 + 2x_2 \le 18$; $x_1 \ge 0, x_2 \ge 0$. **(12 Marks)**

70 (72)

6 a. A department has five employees with five jobs to be performed. The time (in hours) each men will take to perform each job is given in the effectiveness matrix.

		Employees				
		I	II	III	IV	V
	A	10	5	13	15	16
	B	3	9	18	13	6
Jobs	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be allocated? One per employee, so as to minimize the total man hours. Use the Hungarian method. **(10 Marks)**

b. The following table shows all the necessary information on the availability of supply to each warehouse, the requirement of each market and unit transportation cost (in ₹) from each warehouse to each market.

		Market				
		P	Q	R	S	Supply
	A	6	3	5	4	22
Warehouse	B	5	9	2	7	15
	C	5	7	8	6	8
	Demand	7	12	17	9	45

The shipping clerk has worked out the following schedule from experience. 12 units from A to Q, 1 unit from A to R, 8 units from A to S, 15 units from B to R, 7 units from C to P and 1 unit from C to R.

i) Check and see if the clerk has the optimal schedule.

ii) Find the optimal schedule and minimum total transport cost. **(10 Marks)**

7 a. Solve the game whose pay-off matrix to the player A is given in the table. **(10 Marks)**

		B		
		I	II	III
	I	1	7	2
A	II	6	2	7
	III	5	2	6

b. What is a decision tree? How a decision tree is constructed? Raman Industries Ltd. has a new product which they expect has great potential. At the moment they have two courses of action open to them. S_1 = To test the market and S_2 = To drop the product. If they test it, it will cost ₹50,000 and the response could be positive or negative with probabilities of 0.70 and 0.30 respectively. If it is positive, they could either market it with full scale or drop the product. If they market with full scale, then the result might be low, medium or high demand and the respective net pay-offs would be ₹100000, ₹100000 or ₹500000. These outcomes have probabilities of 0.25, 0.55 and 0.20 respectively. If the result of the test marketing is negative, they have decided to drop the product. If at any point, they drop the product, there is a net gain of ₹25,000 from the sale of scrap. All financial values have been discounted to the present. Draw a decision tree for the problem and indicate the most preferred decision. **(10 Marks)**

8 a. Consider the following Fig.Q8(a), where the dashed lines represent the potential links that could be inserted into the network and the number next to each dashed line represents the cost associated with inserting that particular link.

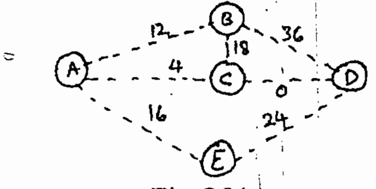

Fig.Q8(a)

Constraint 1 : No more than one of the three links AB, BC and AE can be included

Constraint 2 : Link AB can be included only of link BD also included.

Starting with the initial solution where the inserted links are AB, AC, AE and CD, apply the basic Tabu search algorithm to find the best solution. **(10 Marks)**

b. Write short notes on: i) Simulated annealing technique ii) Genetic algorithm. **(10 Marks)**

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06CS661

Sixth Semester B.E. Degree Examination, June-July 2009

Operations Research

Time: 3 hrs. Max. Marks:100

Note: *Answer any FIVE full questions, selecting at least TWO questions from each part.*

PART – A

1 a. Define : i) Feasible solution ii) Feasible region iii) Optimal solution **(06 Marks)**

b. A manufacturer produces three models I, II, III of certain product using raw materials A and B. The following table gives the data for the problem :

Raw material	Requirements per unit			Availability
	I	II	III	
A	2	3	5	4000
B	4	2	7	6000
Minimum demand	200	200	150	-
Profit per unit (Rs)	30	20	50	-

Formulate the problem as a linear program model. **(07 Marks)**

c. Using graphical method solve the LPP.

Maximize $Z = 5x_1 + 4x_2$

Subject to $6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$-x_1 + x_2 \leq 1$

$x_2 \leq 2, \quad x_1, x_2 \geq 0$ **(07 Marks)**

2 a. Define slack variable and surplus variable. **(04 Marks)**

b. Find all the basic solutions of the following system of equations identifying in each case the basic and non basic variables.

$2x_1 + x_2 + 4x_3 = 11, \quad 3x_1 + x_2 + 5x_3 = 14$ **(06 Marks)**

c. Using simplex method of tabular form solve the LPP.

Maximize $Z = 4x_1 + 3x_2 + 6x_3$

Subject to $2x_1 + 3x_2 + 2x_3 \leq 440$

$4x_1 + 3x_3 \leq 470$

$2x_1 + 5x_2 \leq 430$

$x_1, x_2, x_3 \geq 0$ **(10 Marks)**

3 a. Using two-phase method solve the LPP.

Minimize $Z = 7.5x_1 - 3x_2$

Subject to $3x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$ **(10 Marks)**

b. Using Big-M method solve the CPP.

Maximize $Z = 2x_1 + x_2$

Subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$ **(10 Marks)**

4 a. Use Revised Simplex Method to solve the LPP.

Maximize $Z = 3x_1 + 5x_2$

Subject to $2x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

(10 Mar

b. Explain : i) Weak duality property ii) Strong duality property

iii) Complementary solutions property

(06 Mar

c. Write the dual of the following :

i) Maximize $Z = 6x_1 + 10x_2$

Subject to $x_1 \leq 14$

$x_2 \leq 16$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

ii) Maximize $Z = \begin{pmatrix} 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Subject to $\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 20 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(04 Mark

PART – B

5 a. In Parametric Linear Programming explain about :

i) Systematic changes in the Cj parameters.

ii) Systematic changes in the bj parameters.

(08 Mark

b. Using dual simplex method solve the LPP.

Maximize $Z = -3x_1 - 2x_2$

Subject to $x_1 + x_2 \geq 1$

$x_1 + x_2 \geq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \geq 3$

$x_1, x_2 \geq 0$

(12 Mark

6 a. The transporation costs per truck load of cement (in hundreds of rupees) from each plant t each project site are as follows :

Project sites

		1	2	3	4	
Factories	1	2	3	11	7	6 Supply
	2	1	0	6	1	1
	3	5	8	15	9	10
		7	5	3	2	17

Demand

Determine the optimal distribution for the company so as to minimize the tot transportation cost.

(12 Marks

b. Four jobs are to be done on four different machines. The cost (in rupees) of producing i^{th} jol on the j^{th} machine is given below :

59(72) 1 7

		Machines			
		M_1	M_2	M_3	M_4
Jobs	J_1	15	11	13	15
	J_2	17	12	12	13
	J_3	14	15	10	14
	J_4	16	13	11	17

Assign the jobs to different machines so as to minimize the total cost. **(08 Marks)**

7 a. Solve the game whose payoff matrix to the player A is given below :

		B		
		I	II	III
A	I	1	7	2
	II	6	2	7
	III	5	2	6

(10 Marks)

b. Solve the following (2 x 3) game graphically.

			Y_1	Y_2	Y_3
			I	II	III
A	x_1	I	1	3	11
	$1 - x_1$	II	8	5	2

(10 Marks)

8 a. Use Tabu Search algorithm to find the optimal solution of **(08 Marks)**

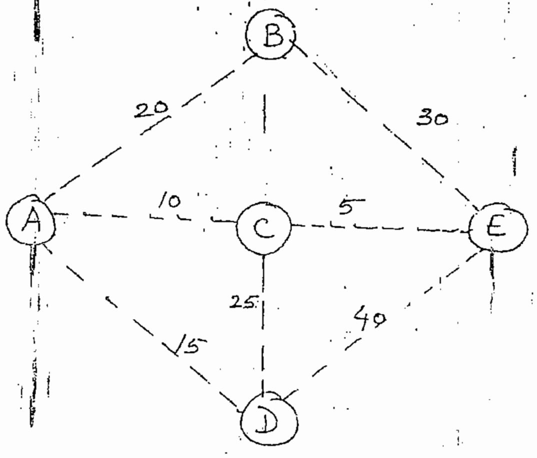

Fig. Q8 (a)

b. Give note on outline of a Basic Simulated Annealing Algorithm. **(06 Marks)**

c. Give note on outline of a Basic Genetic Algorithm. **(06 Marks)**

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ClassRoom
Assignments

Divya

OPERATIONS RESEARCH

ASSIGNMENT

HUNGARIAN METHOD

Step 1 : Subtract the minimum of each row from all the elements of the respective rows

Step 2 : Subtract the minimum of each column from all the elements of the respective columns.

Step 3 : Examine the rows successively until a row-wise exactly single zero is found.
Mark this zero (☐) to make assignment.
Cross out (×) all zeros in the column appropriately
Repeat the procedure for columns also.

Step 4 : If assignment is complete, then find the total assignment cost. If the assignment is not complete, move to step 5.

Step 5 : In this step, we draw minimum number of lines to cover all zeros using the following procedure.

(i) Tick the rows (✓) that donot have marked 0's or
(ii) Tick the columns (✓) having marked 0's
Crossed 0's
(iii) Tick the rows (✓) having marked 0's on the ticked columns.
(iv) Repeat steps (ii) & (iii) until the chain of ticking is complete.
(v) Draw the lines through unticked rows & ticked columns.

Step 6 : Determine the smallest element which is not covered by these lines.

Subtract this minimum element from all uncovered elements and add the same element to the cells where horizontal and vertical lines intersect

Step 7: Repeat steps (iii), (iv), (v) & (vi) until the assignment is complete.

ASSIGNMENT PROBLEM

	R_1	R_2	R_3	R_4
C_1	9	14	19	15
C_2	7	17	20	19
C_3	9	18	21	18
C_4	10	12	18	19
C_5	10	15	21	16

Since the given problem has 5 ~~col~~ rows and 4 columns, a dummy column R_5 is introduced to make the problem balanced.

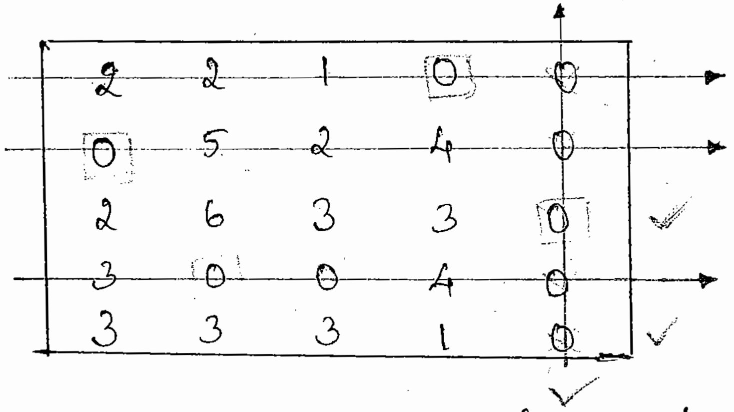
Step 1 ⓐ Row Subtraction

	R_1	R_2	R_3	R_4	R_5
C_1	9	14	19	15	0
C_2	7	17	20	19	0
C_3	9	18	21	18	0
C_4	10	12	18	19	0
C_5	10	15	21	16	0

Step 1 ⓑ Column Subtraction

	R1	R2	R3	R4	R5
C1	2	2	1	0	0
C2	0	5	2	4	0
C3	2	6	3	3	0
C4	3	0	0	4	0
C5	3	3	3	1	0

Step 2 : Draw minimum straight lines to cover all zeros

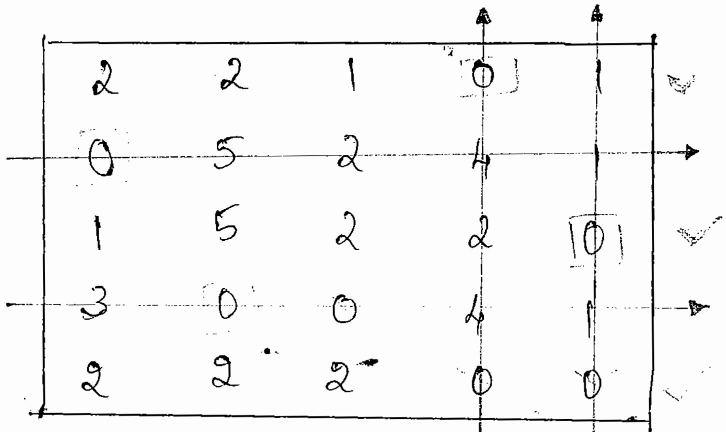


2	2	1	0	0	
0	5	2	4	0	
2	6	3	3	0	✓
3	0	0	4	0	
3	3	3	1	0	✓
				✓	

Step 3 : Smallest uncovered number is then subtracted from uncovered numbers added to numbers at intersection of two lines.

2	2	1	0	1
0	5	2	4	1
1	5	2	2	0
3	0	0	4	1
2	2	2	0	0

Step 4 :



2	2	1	0	1	✓
0	5	2	4	1	
1	5	2	2	0	✓
3	0	0	4	1	
2	2	2	0	0	✓

Step 5:

$$\begin{array}{|ccccc|}
1 & 1 & 0 & 0 & 1 \\
0 & 5 & 2 & 5 & 2 \\
0 & 4 & 1 & 2 & 0 \\
3 & 0 & 0 & 5 & 2 \\
1 & 1 & 1 & 0 & 0
\end{array}$$

Step 6:

$$\begin{array}{c|ccccc|}
 & R_1 & R_2 & R_3 & R_4 & R_5 \\
\hline
C_1 & 1 & 1 & \boxed{0} & \xcancel{0} & 1 \\
C_2 & \boxed{0} & 5 & 2 & 5 & 2 \\
C_3 & \xcancel{0} & 4 & 1 & 2 & \boxed{0} \\
C_4 & 3 & \boxed{0} & \xcancel{0} & 5 & 2 \\
C_5 & 1 & 1 & 1 & \boxed{0} & \xcancel{0} \\
\hline
\end{array}$$

All rows and columns have single allocation and hence optimality criteria is satisfied.

Thus allotments are as follows

$R_1 \xrightarrow{7} C_2$,

$R_2 \xrightarrow{12} C_4$,

$R_3 \xrightarrow{19} C_1$,

$R_4 \xrightarrow{16} C_5$,

$R_5 \xrightarrow{0} C_3$

Total = $\underline{\underline{54}}$

TRAVELLING SALESMAN PROBLEM :

[SENSITIVE ANALYSIS IN ASSIGNMENT]

A salesman estimates that the following would be the cost on his route, visiting the six cities as shown in the table below :

To City

From City	1	2	3	4	5	6
1	∞	20	23	27	29	34
2	21	∞	19	26	31	24
3	26	28	∞	15	36	26
4	25	16	25	∞	23	18
5	23	40	23	31	∞	10
6	27	18	12	35	16	∞

The salesman can visit each of the cities once and only once. Determine the optimum sequence he should follow to minimize the total distance travelled. What is the total distance travelled ?

Solution :

step 1 → Apply usual assignment algorithm

	1	2	3	4	5	6
1	∞	0	3	7	9	14
2	2	∞	0	7	12	5
3	11	13	∞	0	21	11
4	9	0	9	∞	7	2
5	13	30	13	21	∞	0
6	15	6	0	23	4	∞

Row Subtraction

Column Subtraction

	1	2	3	4	5	6
1	∞	0	3	7	5	14
2	0	∞	0	7	8	5
3	9	13	∞	0	17	11
4	7	0	9	∞	3	2
5	11	30	13	21	∞	0
6	13	6	0	23	0	∞

Draw lines through unticked rows & ticked columns

$$\begin{array}{cccccc|c}
\infty & \boxed{0}\checkmark & 3 & 7 & 5 & 14 & \checkmark \\
\boxed{0} & \infty & \cancel{0} & 7 & 8 & 5 & \\
9 & 13 & \infty & \boxed{0}\checkmark & 17 & 11 & \checkmark \\
7 & \cancel{0} & 9 & \infty & 3 & 2 & \checkmark \\
11 & 30 & 13 & 21 & \infty & \boxed{0} & \checkmark \\
13 & 6 & \boxed{0} & 23 & \cancel{0} & \infty & \\
\hline
\checkmark & \checkmark & & \checkmark & & \checkmark &
\end{array}$$

$R_4 - C_5$

$$\begin{array}{cccccc}
\infty & 0 & 3 & 7 & 5 & 14 \\
0 & \infty & 0 & 7 & 8 & 5 \\
9 & 13 & \infty & 0 & 17 & 11 \\
7 & 0 & 9 & \infty & 3 & 2 \\
11 & 30 & 13 & 21 & \infty & 0 \\
13 & 6 & 0 & 23 & 0 & \infty
\end{array}$$

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \infty & \cancel{0} & \boxed{0} & 7 & 2 & 14 \\ 2 & \boxed{0} & \infty & \cancel{0} & 10 & 8 & 8 \\ 3 & 6 & 13 & \infty & \boxed{0} & 14 & 11 \\ 4 & 4 & \boxed{0} & 6 & \infty & \cancel{0} & 2 \\ 5 & 8 & 30 & 10 & 21 & \infty & \boxed{0} \\ 6 & 13 & 9 & \cancel{0} & 26 & \boxed{0} & \infty \end{array}$$

This table gives the optimum assignment solution

$1 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1$

and $5 \rightarrow 6 \rightarrow 5$

with minimum distance $= 23 + 21 + 15 + 16 + 10 + 16$

$= \underline{\underline{101}}$

But this table doesnot provide solution to the travelling salesman problem, as it is not allowed to go from city 2 to 1 without visiting the cities 5 & 6.

Step 2: Once and only once, we try to find the 'next best' solution which satisfies the additional restriction.

The smallest element other than zero is 2. So, we try to bring 2 into the solution. Since the element 2 occurs at 2 places, we shall consider both the cases separately until the acceptable solution is attained.

We start making assignment with the cell (1, 5) having the next minimum element 2 instead of 0 in cell (1, 3). After making this assignment, we observe that no other assignment can be made in the first row and fifth column.

The resulting feasible solution will be

$1 \rightarrow 5 \rightarrow 6 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1$.

The assigned elements for this solution are marked in the following table. The cost corresponding to this feasible solution is 2.

	1	2	3	4	5	6
1	∞	$\cancel{0}$	$\cancel{0}$	7	$\boxed{2}$	14
2	$\boxed{0}$	∞	$\cancel{0}$	10	8	8
3	-6	13	∞	$\boxed{0}$	14	11
4	4	$\boxed{0}$	6	∞	$\cancel{0}$	2
5	8	30	10	21	∞	$\boxed{0}$
6	13	29	$\boxed{0}$	26	$\cancel{0}$	∞

Total cost $= 29 + 21 + 15 + 16 + 10 + 12 = \underline{\underline{103}}$

Find the Initial Basic Feasible solution using NWC

Also find the Optimal Solution

	W_1	W_2	W_3	W_4	W_5	SUPPLY
F_1	4	3	1	2	6	40
F_2	5	2	3	4	5	30
F_3	3	5	6	3	2	20
F_4	2	4	4	5	3	10
DEMAND	30	30	15	20	5	

Solution: Step 1: NORTH-WEST CORNER METHOD

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	30 4	10 3	1	2	6	~~40~~ ~~10~~ 0
F_2	5	20 2	10 3	4	5	~~30~~ ~~10~~ 0
F_3	3	5	5 6	15 3	2	~~20~~ ~~15~~ 0
F_4	2	4	4	5 5	5 3	~~10~~ ~~5~~ 0
Demand	~~30~~ 0	~~30~~ 20 0	~~15~~ ~~5~~ 0	~~20~~ ~~5~~ 0	~~5~~ 0	

$$\text{Supply} = 40 + 30 + 20 + 10 = 100$$

$$\text{Demand} = 30 + 30 + 15 + 20 + 5 = 100$$

Since Supply = Demand, the given problem is balanced

$$\text{Transportation Cost} = (30\times4) + (10\times3) + (20\times2) + (10\times3) + (5\times6) + (15\times3) + (5\times5) + (5\times3)$$

$$= 335$$

Step 2 : We shall check whether the current solution is optimal or not.

	W1	W2	W3	W4	W5	
F1	(30) 4	(10) 3	1	2	6	$u_1 = 0$
F2	5	(20) 2	(10) 3	4	5	$u_2 = -1$
F3	3	5	(5) 6	(15) 3	2	$u_3 = 2$
F4	2	4	4	(5) 5	(5) 3	$u_4 = 4$
	$v_1 = 4$	$v_2 = 3$	$v_3 = 4$	$v_4 = 1$	$v_5 = -1$	

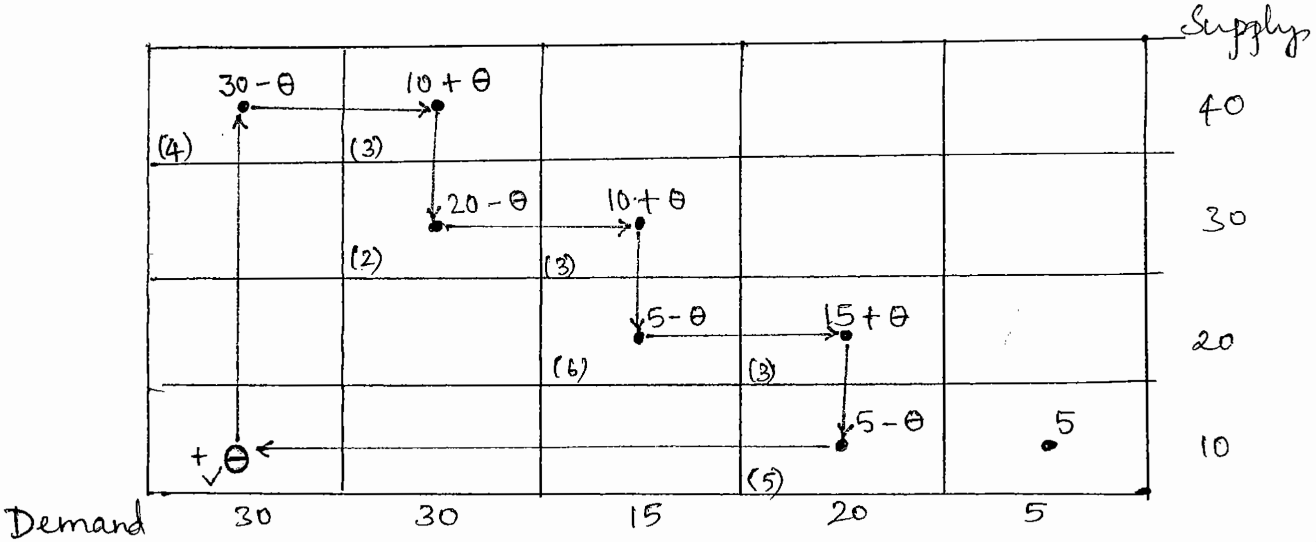
→ Since all the rows have equal number of allocations, let us arbitrarily consider first row & set $u_1 = 0$.

→ We shall find $u_2, u_3, u_4, v_1, v_2, v_3, v_4, v_5$ with the help of the cells where there are allocations.

→ Find $u_i + v_j$ for the cells without allocations (we write these entries at bottom right corner)

→ Find $d_{ij} = c_{ij} - (u_i + v_j)$ for the cells without allocations

	W1	W2	W3	W4	W5	
F1	●	●	-3 1 4	1 2 1	7 6 -1	$u_1 = 0$
F2	2 5 3	●	●	4 4 0	7 5 -2	$u_2 = -1$
F3	-3 3 6	0 5 5	●	●	1 2 1	$u_3 = 2$
F4	-6 2 8	-3 4 7	-4 4 8	●	●	$u_4 = 4$
	$v_1 = 4$	$v_2 = 3$	$v_3 = 4$	$v_4 = 1$	$v_5 = -1$	



Since, the largest negative cell evaluation is $d_{41} = -6$, allocate as much as possible to this cell (4,1). This necessitates shifting of 5 units to this cell (4,1) as directed by the closed loop in the above table.

Here, the maximum value of θ is obtained by usual rule:

$\min[30-\theta, 20-\theta, 5-\theta, 5-\theta] = 0$ i.e. $5-\theta = 0 \Rightarrow \theta = 5$ units

Revised solution:

					Supply
25(4)	15(3)				40
	15(2)	15(3)			30
		$\overset{*}{0}(6)$	20(3)		20
5(2)			$\overset{*}{0}(5)$	5(3)	10
Demand 30	30	15	20	5	

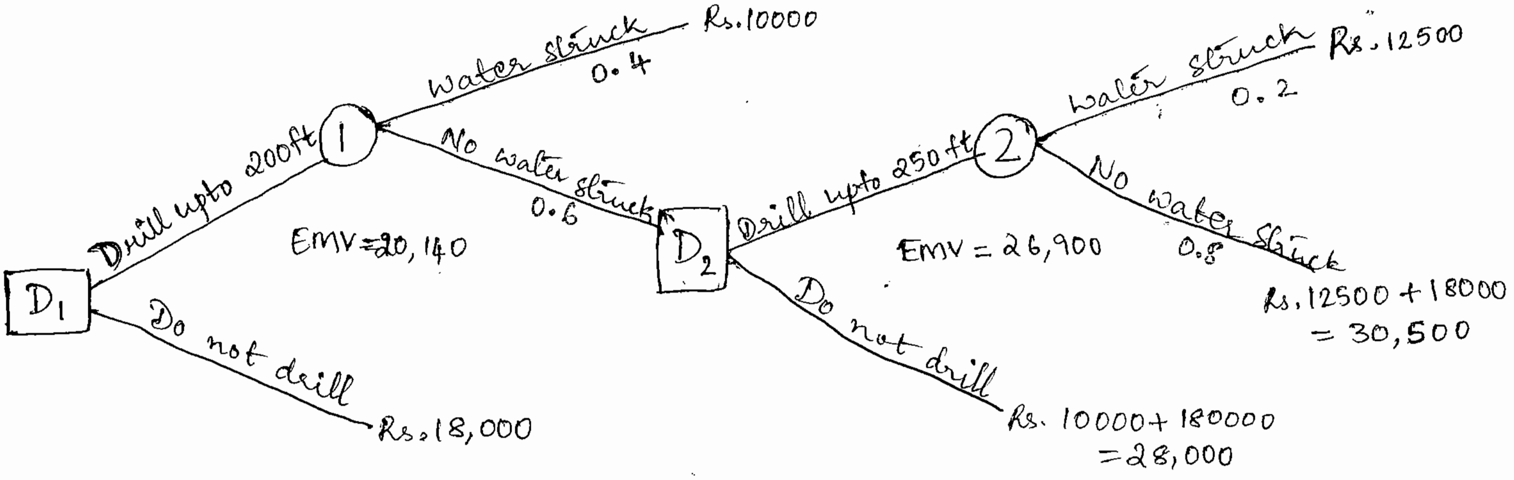
In this solution, the number of allocations becomes less than m+n−1, on account of simultaneous vacation of two cells (3,3), (4,4) as indicated by *.

Hence this is a degenerate solution.

DECISION TREE

Mr. Sinha had to decide whether or not to drill a well on his farm. In his village, only 40% of the wells drilled were successful at 200 feet of depth. Some of the farmers who didnot get water at 200 feet, drilled further upto 250 feet but only 20% struck water at 250 feet. Cost of drilling is Rs. 50 per foot. Mr. Sinha estimated that he would pay Rs. 18,000 during a 5-year period in the present value terms, if he continues to buy water from the neighbour rather than go for the well which would have a life of 5 years. Mr. Sinha has three decisions to make: ⓐ should he drill upto 200 feet and ⓑ if no water is found at 200 feet, should he drill upto 250 feet? ⓒ should he continue to buy water from his neighbour?

Solution:



The decision tree representing possible courses of action and states of nature are shown in above figure.

In order to analysis the tree, we start working backward from the end nodes.

The cost associated with each outcome is written on the decision tree.

$EMV(2) = 0.2 \times 12,500 + 0.8 \times 30,500 = 2500 + 24,400 = \underline{26,900}$

$EMV(D_2)$ = Rs. 26,900 (Minimum of two values ; Rs. 26,900 & Rs. 28,000)

$EMV(1) =$ ~~Rs~~ $0.4 \times 10000 + 0.6 \times 26,900 = 4000 + 16140 = \underline{20,140}$

$EMV(D_1)$ = Rs. 18000 (Minimum of two values ; Rs. 20,140 & Rs. 18000)

∴ Thus the optimal (least cost) course of action for Mr. Sinha is not to drill ~~the~~ well and pay Rs. 18,000 for water to his neighbour for five years.

DECISION ANALYSIS PROBLEM

A farmer is attempting to decide which of three crops he should plant on his 100 acre farm. The profit from each crop is strongly dependent on the rainfall during the growing season. He has categorised the amount of rainfall as substantial, moderate or light. He estimates his profit for each crop as shown below:

Rainfall	Estimated Profit (in Rs.)		
	Crop A	Crop B	Crop C
Substantial	7,000	2,500	4,000
Moderate	3,500	3,500	4,000
Light	1,000	4,000	3,000

Based on the weather in previous seasons and the current projection for the coming season, he estimates the probability of substantial rainfall as 0.2, that of moderate rainfall as 0.3 and that of light rainfall as 0.5. Furthermore, services of forecasters could be employed to provide a detailed survey of current rainfall prospects as shown in Table

Rainfall	Estimated Profit (in Rs.)		
	Crop A	Crop B	Crop C
Substantial	0.70	0.25	0.05
Moderate	0.30	0.60	0.10
Light	0.10	0.20	0.70

(a) From the available data, determine the optimal decision as to which crop to plant.

(b) Determine whether it would be economical for the farmer to hire the services of a forecaster.

Solution : (a)

$E(\text{Crop } A) = (0.2 \times 7000) + (0.3 \times 3500) + (0.5 \times 1000) = \underline{\underline{2950}}$

$E(\text{Crop } B) = (0.2 \times 2500) + (0.3 \times 3500) + (0.5 \times 4000) = \underline{\underline{3550}}$

$E(\text{Crop } C) = (0.2 \times 4000) + (0.3 \times 4000) + (0.5 \times 3000) = \underline{\underline{3500}}$

The maximum EMV is Rs. 3550. Therefore, optimal course of action is plant Crop B. However, it would make no sense to plant more than one kind of crop because maximum EMV is obtained by planting all 100 acres with crop B.

(b)

Let B_i $(i = 1, 2, 3)$ denote the outcome forecast for 'substantial rainfall', 'moderate rainfall' and 'light rainfall' respectively. The likelihood values are given below:

$$EPPI = 0.2(7000) + 0.3(4000) + 0.5(4{,}000) = \underline{Rs. 4{,}600}$$

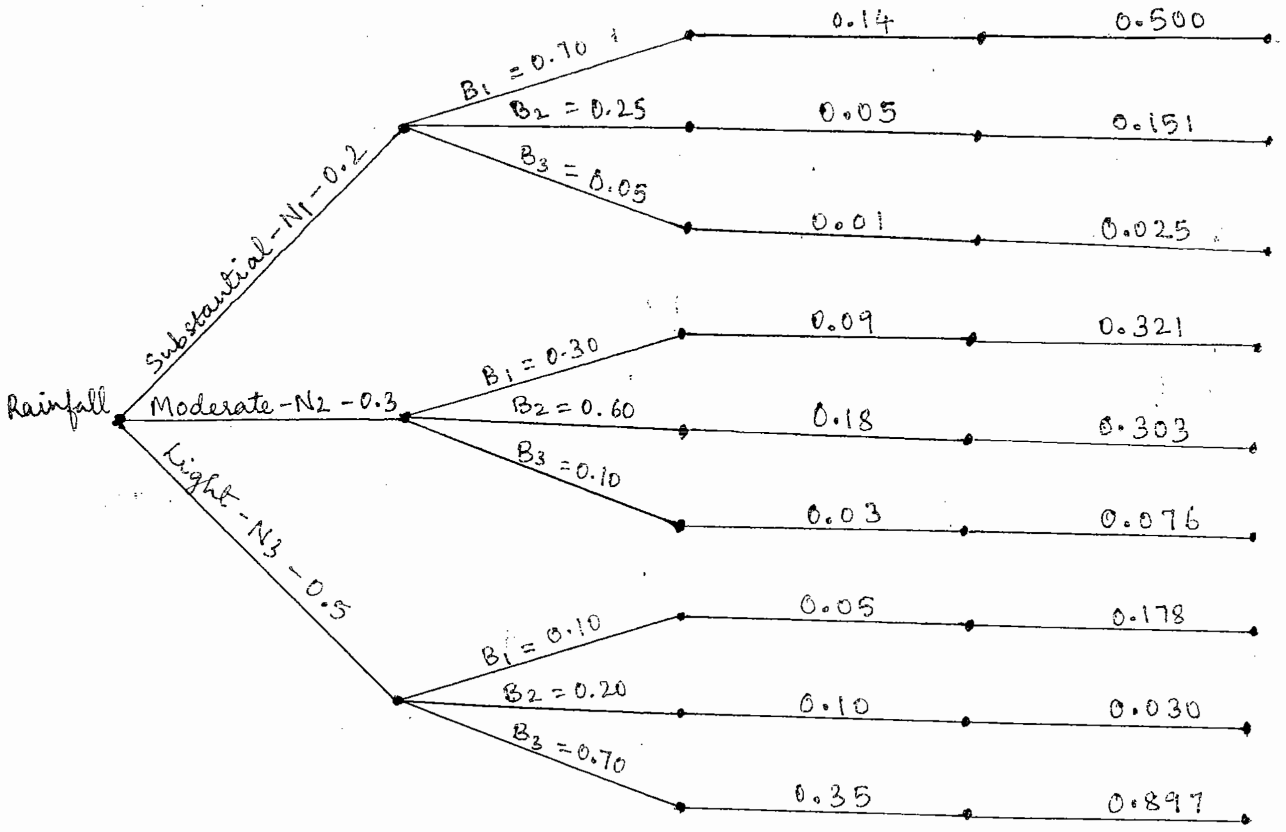
Thus, we have $EVPI = EPPI - EMV^* = 4{,}600 - 3550 = \underline{Rs. 1{,}050}$

Prior Probabilities

States of Nature	Prior probability	Outcomes (B_i)	Conditional probability $P(B_i \mid N_i)$	Joint Probability $P(B_i \cap N_i) = P(N_i)\,P(B_i \mid N_i)$		
N_1	0.2	B_1	0.70	0.14	–	–
		B_2	0.25	–	0.05	–
		B_3	0.05	–	–	0.01
N2	0.3	B_1	0.30	0.09	–	–
		B_2	0.60	–	0.18	–
		B_3	0.10	–	–	0.03
N3	0.5	B_1	0.10	0.05	–	–
		B_2	0.20	–	0.10	–
		B_3	0.70	–	–	0.35
Marginal Probability				0.28	0.33	0.39

Posterior Probabilities

Outcome (B_i)	Probability $P(B_i)$	States of Nature (N_i)	Posterior Probability $P(N_i \mid B_i) = P(N_i \cap B_i) / P(B_i)$
B_1	0.28	N_1	0.14 / 0.28 = 0.500
		N_2	0.09/0.28 = 0.321
		N_3	0.05/0.28 = 0.178
B_2	0.33	N_1	0.05 / 0.33 = 0.151
		N_2	0.18 / 0.33 = 0.303
		N_3	0.10 / 0.33 = 0.030
B_3	0.39	N_1	0.01 / 0.39 = 0.025
		N_2	0.03 / 0.39 = 0.076
		N_3	0.35 / 0.39 = 0.897



VAM technique :

	C_1	C_2	C_3	C_4	C_5	Supply
D_1	2	11	10	3	7	4
D_2	1	4	7	2	1	8
D_3	3	9	4	8	12	9
Demand	3	3	4	5	6	

Solution :

Supply = 4 + 8 + 9 = 21

Demand = 3+3+4+5+6 = 21

Since supply = demand, ~~the~~ problem is balanced.

	C_1	C_2	C_3	C_4	C_5	Supply	Row Penalties
D_1	2	11	10	•4 3	7	~~4~~ 0	1 1 1 1 – –
D_2	1	•2 4	7	2	•6 1	~~8~~ ~~2~~ 0	0 1 – – – –
D_3	•3 3	•1 9	•4 4	•1 8	12	~~9~~ ~~5~~ ~~4~~ ~~3~~ 0	1 1 1 5 5 5
Demand	~~3~~ 0	~~3~~ ~~1~~ 0	~~4~~ 0	~~5~~ ~~1~~ 0	~~6~~ 0		

Column Penalties				
1	5	3	1	6
1	5	3	1	–
1	2	6	5	–
1	2	–	5	–
3	9	–	8	–
3	–	–	8	–

Step 1 : Highest penalty is 6. and it corresponds to 5th column
Least cost is (2,5) = 1. $x_{25} = \min(8, 6) = 6$

Step 2 : Highest penalty is 5 and it corresponds to 2nd column
Least cost is (2,2) = 4. $x_{22} = \min(2, 3) = 2$

Step 3 : Highest penalty is 6 and it corresponds to 3rd column
Least cost is (3,3) = 4. $x_{33} = \min(9, 4) = 4$

Step 4 : Highest penalty is 5 and it corresponds to 4th column least cost is (1, 4) = 3. $x_{14} = \min(4,5) = 4$

Step 5 : Highest penalty is 9 and it corresponds to 2nd column $\therefore x_{32} = \min(5,1) = 1$.

Step 6 : Highest penalty is 8 and it corresponds to 4th column $x_{34} = \min(4,1) = 1$

Step 7 : Allocate 3 to (3,1)

Transportation cost = $(4\times3) + (2\times4) + (6\times1) + (3\times3) + (1\times9)$
$+ (4\times4) + (1\times8)$.

$= \underline{\underline{68}}$

SADDLE POINT.

	I	II	III	IV
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	-5	4	11	0

Solution :

Ⓑ

Ⓐ

	I	II	III	IV	Row min
I	20	15	12	35	12
II	25	14	8	10	8
III	40	2	10	5	2
IV	-5	4	11	0	-5
Col max	40	15	12	35	

max-min values = max (12, 8, 2, -5) = 12

min-max values = min (40, 12, 15, 35) = 12

Since $\underline{V}$ and $\overline{V}$ are equal, the given pay-off matrix has a saddle point.

Optimal Strategy for player A → I

optimal strategy for player B → III

Value of the game = 12

DOMINANCE STRATEGY

Ⓑ

	1	3	2	7	4
Ⓐ	3	4	1	5	6
	6	5	7	6	5
	2	0	6	3	1

Solution:

	b_1	b_2	b_3	b_4	b_5
a_1	1	3	2	7	4
a_2	3	4	1	5	6
a_3	6	5	7	6	5
a_4	2	0	6	3	1

We note that $a_4 \leq a_3$, delete a_4

$b_4 \geq b_2$. delete b_4 .

$b_5 \geq b_2$. delete b_5

$a_1 \leq a_3$ delete a_1

$a_2 \leq a_3$ delete a_2

$b_3 \geq b_2$ delete b_3

$b_1 \geq b_2$ delete b_1

Further reduction of the game by dominance rule is not possible. Hence the given game is reduced to the following 1×1 game

$$\begin{array}{cc} & b_2 \\ a_3 & \boxed{5} \end{array}$$

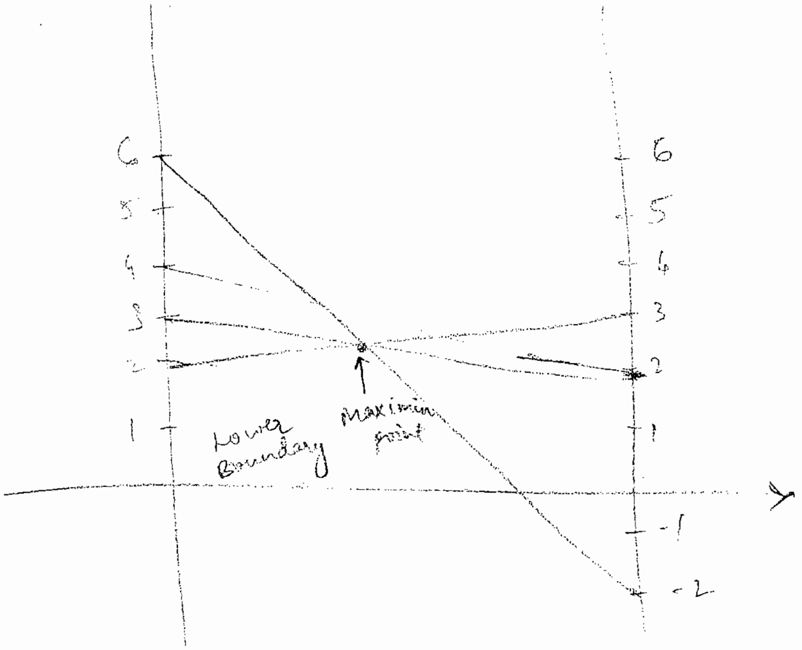
Optimal Strategy for A = a_3
optimal strategy for B = b_2

∴ Value of the game = $\underline{\underline{5}}$

Since the value of the game is positive, it is advantageous to player A.

ALGEBRAIC METHOD :

	B1	B2	B3	B4
A1	2	2	3	−2
A2	4	3	2	6



Eq ② & ④

This game doesnot have saddle point. Thus, the player A's expected payoffs corresponding to the player B's pure strategies are

B's pure strategies	A's expected Pay off $E(x_1)$
I	$E(x_1) = 2x_1 + 4(1-x_1)$
II	$E(x_1) = 2x_1 + 3(1-x_1)$
III	$E(x_1) = 3x_1 + 2(1-x_1)$
IV	$E(x_1) = -2x_1 + 6(1-x_1)$

Maximin occurs at $x_1 = 1/2$

Eqn II $\rightarrow E(x_1) = 2x_1 + 3 - 3x_1 = -x_1 + 3$

Eqn IV $\rightarrow E(x_1) = -2x_1 + 6 - 6x_1 = -8x_1 + 6$

OPERATIONS RESEARCH

TWO PHASE METHOD

Minimize $z = \frac{15}{2} x_1 - 3x_2$

ST $3x_1 - x_2 - x_3 \geqslant 3$

$x_1 - x_2 + x_3 \geqslant 2$

and $x_1, x_2, x_3 \geqslant 0$

Solution

Convert the objective function into the maximization form:

maximize $z' = \frac{-15}{2} x_1 + 3x_2$

Introducing the surplus variables $S_1 \geqslant 0$ and $S_2 \geqslant 0$ and artificial variables $a_1 \geqslant 0$, $a_2 \geqslant 0$, the constraints of the given problem becomes:

$$3x_1 - x_2 - x_3 - S_1 + a_1 = 3$$

$$x_1 - x_2 + x_3 - S_2 + a_2 = 2$$

$$x_1, x_2, x_3, S_1, S_2, a_1, a_2 \geqslant 0$$

PHASE I

Assigning a cost -1 to artificial variables a_1 and a_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes:

$$\text{Max } z'^{*} = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1a_1 - 1a_2$$

subject to the above given constraints

Now, apply simplex method in usual manner.

			0	0	0	0	0	-1	-1		
BV	C_B	X_B	x_1	x_2	x_3	S_1	S_2	a_1	a_2	Ratios	
a_1	-1	3	$\boxed{3}$	-1	-1	-1	0	1	0	$3/3 = 1$	← PR
a_2	-1	2	1	-1	1	0	-1	0	1	$2/1 = 2$	
	$Z'^* = -5$		-4	2	0	1	1	0	0		

↑
PC

Pivot Element = 3
Entering variable = x_1
Leaving variable = a_1

Apply $R_1' = R_1 \div 3$

$R_2' = R_2 -$ [illegible]R_1'

			0	0	0	0	0	-1	-1		
BV	C_B	X_B	x_1	x_2	x_3	S_1	S_2	a_1	a_2	Ratios	
x_1	0	1	1	$-1/3$	$-1/3$	$-1/3$	0	X	0	IGNORE	
a_2	-1	1	0	$-2/3$	$\boxed{4/3}$	$1/3$	-1	X	1	$\frac{1}{4/3} = 3/4$	← PR
	$Z'^* = -1$		0	$2/3$	$-4/3$	$-1/3$	1	X	0		

↑
PC

Pivot Element = $4/3$
Entering variable = x_3
Leaving variable = a_2

Apply $R_2' = \frac{3}{4} R_2$

$R_1' = R_1 + \frac{1}{3} R_2'$

			0	0	0	0	0	-1	-1	
BV	C_B	X_B	x_1	x_2	x_3	S_1	S_2	a_1	a_2	Ratios
x_1	0	5/4	1	-1/2	0	-1/4	-1/4	X	X	
x_3	0	3/4	0	-1/2	1	1/4	-3/4	X	X	
	$Z'^* = 0$		0	0	0	0	0	X	X	

We note that max ~~p~~ $z'^* = 0$ and no artificial variable appears in the optimum basis.

∴ The given LPP has a solution.

PHASE 2

Consider the final simplex table obtained at the end of Phase 1. Assign the actual cost of the OF.

$$\max z' = -15/2\, x_1 + 3x_2 + 0S_1 + 0S_2$$

Now apply simplex method in usual manner.

			-15/2	3	0	0	0	
BV	C_B	X_B	x_1	x_2	x_3	S_1	S_2	Ratios
x_1	-15/2	5/4	1	-1/2	0	-1/4	-1/4	
x_3	0	3/4	0	-1/2	1	1/4	-3/4	
	$Z'^* =$	-75/8	0	3/4	0	15/8	15/8	

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been obtained. Hence optimal solution is :

$$\boxed{x_1 = \frac{5}{4},\ x_2 = 0,\ x_3 = \frac{3}{4}} \text{ and } \boxed{\min\ z = \frac{75}{8}}$$

∴ SIMULATED ANNEALING

Solution

Simulated annealing is another widely used metaheuristic that enables the search process to escape from a local optimum

→ Initialization

Start with a feasible initial trial solution.

→ Iteration

Use the move selection rule to select the next trial solution (If none of the immediate neighbours of the current ~~trial~~ solution are accepted, the algorithm is terminated).

Move selection rule states that among all the neighbours of the current trial solution, select one randomly ~~because~~ to become the current candidate to be the next trial solution. Assuming the objective is maximization of the objective function, accept or reject this candidate to be the next trial solution as follows:

If $Z_n \geqslant Z_c$, always accept the candidate.

If $Z_n < Z_c$, accept the candidate with the following probability:

$$\text{Prob}\{\text{acceptance}\} = e^{x} \quad \text{where } x = \frac{Z_n - Z_c}{T}$$

where Z_c = OF value for the current trial solution

Z_n = OF value for the current candidate to be the next trial solution.

T = a parameter that measures the tendency to accept the current candidate to the next trial solⁿ

→ Check the temperature schedule

When the desired number of iterations have been performed at the current value of T, decrease T to the next value in the temperature schedule and resume performing iterations at this next value.

→ Stopping Rule

When the desired number of iterations have been performed at the smallest value of T in the temperature schedule (or when none of the immediate neighbours of the current trial solution are accepted), stop. Accept the best trial solution found at any iteration as the final solution

GENETIC ALGORITHM

Solution

Genetic Algorithm provide a third type of metaheuristic which tends to be particularly effective at exploring various parts of the feasible region and gradually evolving toward the best feasible solutions.

→ Initialization

Start with an initial population of feasible trial solutions, perhaps by generating them randomly. Evaluate the fitness (the value of the objective function) for each member of this current population.

→ Iteration

Use a random process that is biased toward the more fit members of the current population to select some of the members (an even number) to become parents. Pair up the parents randomly and then have each pair of parents give birth to two children (new feasible trial solutions) whose features (genes) are a random mixture of the features of the parents, except for occasional mutations. (Whenever the random mixture of features and any mutations result in an infeasible solution, this is a miscarriage, so the process of attempting to give birth then is repeated until a child is born that corresponds to a feasible solution) Retain the children and enough of the best members of the current population to form the new population of the same size for the next iteration. (Discard the other members of the current population). Evaluate the fitness for each new member (the children) in the new population.

→ Stopping Rule

Use some stopping rule, such as a fixed number of iterations, a fixed amount of CPU time, or a fixed number of consecutive iterations without any improvement in the best trial solution found so far. Use the best trial solution found on any iteration as the final solution.

PRIMAL TO DUAL

WORKING RULE (Only for Reference, so that given any problem, you must be able to convert from primal to dual form)

Step 1: Convert the OF to maximization type (if required)

Step 2: Convert the inequalities of the type $\geq$ to $\leq$ by multiplying both the sides by -1.

Step 3: If the constraint has an equality, then it is replaced by two constraints involving $\leq$ to $\geq$.

Suppose we have $x + 2y = 4$

$$\left.\begin{array}{l}\Rightarrow x + 2y \leq 4 \\ \Rightarrow x + 2y \geq 4\end{array}\right\} \Rightarrow \begin{array}{l} x + 2y \leq 4 \\ -x - 2y \leq -4\end{array}$$

Step 4 : Every unrestricted variable (a variable which can take +ve, -ve or 0 values) is replaced by the difference of two non-negative variables (≥ 0)

Step 5 : Reduce the given LPP to standard form in which all the constraints are of the type $\leq$ and the OF is of maximization type.

Step 6 : We write the dual of the given problem using the following steps:

(i) Transpose the rows and columns of the constraint co-efficients

(ii) Transpose the co-efficients of OF $c_1, c_2, c_3 ---- c_n$ and the right side constraints $b_1, b_2, ----- b_n$

(iii) Change $\leq$ to $\geq$.

(iv) Change max to min

RELATIONSHIP BETWEEN STANDARD PRIMAL PROBLEM AND ITS DUAL PROBLEM (Jan 2010, June 2010, Dec 2010)

Standard Primal	Its Dual
• Objective function, to max Z_p	• Objective function, to min Z_d
• Requirement vector	• Price vector
• Coefficient matrix, A	• Transpose of coefficient matrix, A^T
• Constraints with $\leq$ sign	• Constraints with $\geq$ sign
• Relation	• Variable

ith inequality	ith variable, $w_i \geqslant 0$
ith constraint an equality	ith variable w_i unrestricted
variable	Relation
If ith variable $x_i > 0$	ith relation a strict equality
ith variable x_i, unrestricted in sign	ith constraint a strict equality
If ith slack variable positive	ith variable zero
If ith variable zero	ith surplus variable positive
Finite optimum solution	Finite optimum solution with equal optimal value of OF
Unbounded solution	No solution or unbounded sol^n

Example:

Primal Problem

$$\text{Max} \quad Z_x = 40x_1 + 50x_2$$

subject to

$$2x_1 + 3x_2 \leqslant 3$$

$$8x_1 + 4x_2 \leqslant 5$$

$$x_1, x_2 \geqslant 0$$

Solution:

Dual Problem

$$\text{Min} \quad Z_w = 3w_1 + 5w_2$$

subject to

$$2w_1 + 8w_1 \geqslant 40$$

$$3w_1 + 4w_2 \geqslant 50$$

$$w_1, w_2 \geqslant 0$$

∴ VOGEL'S APPROXIMATION METHOD

	C_1	C_2	C_3	C_4	Supply
R_1	1	2	1	4	30
R_2	3	3	2	1	50
R_3	4	2	5	9	20
Demand	20	40	30	10	

Solution

Supply = 30 + 50 + 20 = 100

Demand = 20 + 40 + 30 + 10 = 100

Since supply = demand, TP is balanced

	C_1	C_2	C_3	C_4	Supply	Row penalties
R_1	.20 1	2	.10 1	4	~~30~~ ~~10~~ 0	0 0 1 1 – –
R_2	3	.20 3	.20 2	.10 1	~~50~~ ~~40~~ ~~20~~ 0	1 1 1 1 1 2
R_3	4	.20 2	5	9	~~20~~ 0	2 2 3 – – –
Demand	~~20~~ 0	~~40~~ ~~20~~ 0	~~30~~ ~~20~~ 0	~~10~~ 0		

Column Penalties	2	0	1	3
	2	0	1	–
	–	0	1	–
	–	1	1	–
	–	3	2	–
	–	–	2	–

Step 1 : Highest penalty is 3 & it corresponds to C_4.
Least cost is $(2,4) = 1$
Allocate $\min(50, 10) = 10$ to $(2,4)$

Step 2 : Highest penalty is 2 & it corresponds to C_1.
Least cost is $(1,1) = 1$
Allocate $\min(30, 20) = 20$ to $(1,1)$

Step 3 : Highest penalty is 3 & it corresponds to R_3
Least cost is $(3,2) = 2$
Allocate $\min(20, 40) = 20$ to $(3,2)$

Step 4 : Highest penalty is 1 & it corresponds to R_1
Least cost is $(1,3) = 1$
Allocate $\min(10, 30) = 10$ to $(1,3)$.

Step 5 : Highest penalty is 3 & it corresponds to C_2.
~~Least cost is (2~~
Allocate $\min(40, 20) = 20$ to $(2,2)$

Step 6 : Allocate 20 to $(2,3)$

Transportation cost is

$$= (20 \times 1) + (10 \times 1) + (20 \times 3) + (20 \times 2) + (10 \times 1) + (20 \times 2)$$

$$= \underline{\underline{180}}$$

Thus we obtain a basic initial feasible solution.

TABU SEARCH – MINIMUM SPANNING TREE (10 marks)
Refer Class Notes if you want to attempt this in internals

Example for Section 5.2

Consider the following model.

Maximize $Z = 4\,x_1 + 3\,x_2 + 6\,x_3,$

subject to

$$3\,x_1 + x_2 + 3\,x_3 \leq 30$$
$$2\,x_1 + 2\,x_2 + 3\,x_3 \leq 40$$

and

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

We introduce x_4 and x_5 as slack variables for the respective constraints. The augmented form of the model then is

Maximize $Z = 4\,x_1 + 3\,x_2 + 6\,x_3,$

subject to

$$3\,x_1 + x_2 + 3\,x_3 + x_4 = 30$$
$$2\,x_1 + 2\,x_2 + 3\,x_3 + x_5 = 40$$

and

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0.$$

Using matrix notation, we have

$$\mathbf{c} = \begin{bmatrix} 4 & 3 & 6 & 0 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Now let us apply the revised simplex method step by step to solve this problem.

Iteration 0:

Since x_4 and x_5 are the initial basic variables,

$$\mathbf{x}_B = \begin{bmatrix} x4 \\ x5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \text{ so } \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}.$$

$$\mathbf{c}_B = \begin{bmatrix} 0, & 0 \end{bmatrix}$$

Iteration 1:

The coefficients in Eq. (0) are

$$\mathbf{c_B B^{-1} A - c} = [-4 \quad -3 \quad 6] \quad \text{for } x_1, x_2, x_3,$$
$$\mathbf{c_B B^{-1}} = [0 \quad 0] \quad \text{for } x_4, x_5.$$

Since -6 is the most negative coefficient, we choose x_3 as the entering basic variable. The coefficients of x_3 in Eqs. (1) and (2) are

$$\mathbf{B}^{-1}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The right-hand side of these equations was identified in Iteration 0 as

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}, \text{ where this gives the value of } \mathbf{x_B} = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}.$$

Applying the minimum ratio test, since (30/3) < (40/3), we choose x_4 as the leaving basic variable. Thus,

$$\mathbf{x_B} = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}.$$

Since the coefficients of x_3 in Eqs. (1) and (2) are $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and Eq. (1) yielded the leaving basic variable,

$$\eta = \begin{bmatrix} 1/3 \\ -3/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1 \end{bmatrix}, \qquad \text{so} \quad \mathbf{B}^{-1} \text{ becomes}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1/3 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

$$c_B = [6, \quad 0] \; .$$

Iteration 2:

The coefficients in Eq. (0) now are

$$\mathbf{c_B B^{-1} A - c} = [2 \quad -1 \quad 0] \quad \text{for } x_1, x_2, x_3,$$

$$\mathbf{c_B B^{-1}} = [2 \quad 0] \quad \text{for } x_4, x_5.$$

Since only x_2 has a negative coefficient, it becomes the entering basic variable. The coefficients of x_2 in Eqs. (1) and (2) are

$$\mathbf{B}^{-1}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1/3 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1/3\\1\end{bmatrix}.$$

The right-hand side of these equations was identified in Iteration 1 as

$$\mathbf{B^{-1}b} = \begin{bmatrix}10\\10\end{bmatrix}, \text{ where this gives the value of } \mathbf{x_B} = \begin{bmatrix}x_3\\x_5\end{bmatrix}.$$

Applying the minimum ratio test, since

$$\frac{10}{1/3} > \frac{10}{1},$$

we choose x_5 as the leaving basic variable. Thus,

$$\mathbf{x_B} = \begin{bmatrix}x_3\\x_2\end{bmatrix}.$$

Since the coefficients of x_2 in Eqs. (1) and (2) are $\begin{bmatrix}1/3\\1\end{bmatrix}$ and Eq. (2) yielded the leaving basic variable,

$$\eta = \begin{bmatrix}-(1/3)/1\\1/1\end{bmatrix} = \begin{bmatrix}-1/3\\1\end{bmatrix}, \quad \text{so } \mathbf{B}^{-1} \text{ becomes}$$

$$\mathbf{B}^{-1} = \begin{bmatrix}1 & -1/3\\0 & 1\end{bmatrix}\begin{bmatrix}1/3 & 0\\-1 & 1\end{bmatrix} = \begin{bmatrix}2/3 & -1/3\\-1 & 1\end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 10 \end{bmatrix}.$$

$\mathbf{c}_B = [6, \quad 3]$.

Optimality Test:

The coefficients in Eq. (0) now are

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [6 \quad 3] \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix} - [4 \quad 3 \quad 6]$$

$$= [1 \quad 0 \quad 0] \qquad \text{for } x_1, x_2, x_3,$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [1 \quad 1] \qquad \text{for } x_4, x_5.$$

Since all these coefficients are nonnegative, the current solution is optimal. This solution is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 20/3 \\ 10 \end{bmatrix},$$

as found in Iteration 2. The other variables are nonbasic variables, so $x_1 = 0$, $x_4 = 0$, and $x_5 = 0$.

$$Z = 4(0) + 3(10) + 6(20/3) = 70.$$

Example for Section 5.3

Consider the following problem.

Maximize $Z = x_1 - x_2 + 2x_3$,

subject to

$$\begin{aligned} x_1 + x_2 + 3x_3 &\leq 15 \\ 2x_1 - x_2 + x_3 &\leq 2 \\ -x_1 + x_2 + x_3 &\leq 4 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Let x_4, x_5, and x_6 denote the slack variables for the respective constraints. After the simplex method is applied, a portion of the final simplex tableau is as follows:

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1				0	$\frac{3}{2}$	$\frac{1}{2}$	
x_4	(1)	0				1	-1	-2	
x_3	(2)	0				0	$\frac{1}{2}$	$\frac{1}{2}$	
x_2	(3)	0				0	$-\frac{1}{2}$	$\frac{1}{2}$	

(a) Use the fundamental insight presented in Sec. 5.3 to identify the missing numbers in the final simplex tableau. Show your calculations. Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

From the coefficients for (x_4, x_5, x_6) in the final simplex tableau, we observe that

$$S^* = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \quad \text{and} \quad y^* = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix}.$$

Thus, by the fundamental insight presented in Sec. 5.3, the constraint coefficients for (x_1, x_2, x_3) in the final tableau are

$$S^*A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1 \\ -3/2 & 1 & 0 \end{bmatrix}.$$

The coefficients in the objective function for (x_1, x_2, x_3) in the final tableau are

$$yA - c = \begin{bmatrix} 0 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 & 0 \end{bmatrix}.$$

The final right-hand side is $\mathbf{S^*b} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, and

$$Z^* = \mathbf{y^*b} = [0 \quad 3/2 \quad 1/2] \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix} = 5.$$

Therefore, the complete final simplex tableau is

Basic Variable	Eq	Coefficient of:							Right Side
		Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	(0)	1	3/2	0	0	0	3/2	1/2	5
x_4	(1)	0	1	0	0	1	-1	-2	5
x_3	(2)	0	1/2	0	1	0	1/2	1/2	3
x_2	(3)	0	-3/2	1	0	0	-1/2	1/2	1

(b) Identify the defining equations for the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

Since the optimal BF solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 1, 3, 5, 0, 0)$, the corresponding CPF solution is $(x_1, x_2, x_3) = (0, 1, 3)$. The nonbasic variables are x_1, x_5, x_6, and these are the indicating variables that indicate that the following constraints hold with equality.

$$\begin{aligned} x_1 & \geq 0 \\ 2x_1 - x_2 + x_3 & \leq 2 \\ -x_1 + x_2 + x_3 & \leq 4 \end{aligned}$$

Therefore, the defining equations for this CPF solution are

$$\begin{aligned} x_1 &= 0, \\ 2x_1 - x_2 + x_3 &= 2, \\ -x_1 + x_2 + x_3 &= 4. \end{aligned}$$

ELECTIVE-I (GROUP A)

OPERATIONS RESEARCH

Subject Code	: 06IS661	IA Marks	: 25
No. of Lecture Hours/Week	: 04	Exam Hours	: 03
Total No. of Lecture Hours	: 52	Exam Marks	:100

PART - A

UNIT - 1

INTRODUCTION, LINEAR PROGRAMMING – 1: Introduction: The origin, nature and impact of OR; Defining the problem and gathering data; Formulating a mathematical model; Deriving solutions from the model; Testing the model; Preparing to apply the model; Implementation.
Introduction to Linear Programming: Prototype example; The linear programming (LP) model.

6 Hours

UNIT - 2

LP – 2, SIMPLEX METHOD – 1: Assumptions of LP; Additional examples.
The essence of the simplex method; Setting up the simplex method; Algebra of the simplex method; The simplex method in tabular form; Tie breaking in the simplex method.

7 Hours

UNIT - 3

SIMPLEX METHOD – 2: Adapting to other model forms; Post optimality analysis; Computer implementation. Foundation of the simplex method.

6 Hours

UNIT - 4

SIMPLEX-METHOD – 2, DUALITY THEORY: The revised simplex method, a fundamental insight.
The essence of duality theory; Economic interpretation of duality. Primal dual relationship; Adapting to other primal forms.

7 Hours

PART - B

UNIT - 5

DUALITY THEORY AND SENSITIVITY ANALYSIS, OTHER ALGORITHMS FOR LP: The role of duality in sensitive analysis; The essence of sensitivity analysis; Applying sensitivity analysis. The dual simplex method; Parametric linear programming; The upper bound technique.

7 Hours

UNIT - 6

TRANSPORTATION AND ASSIGNMENT PROBLEMS: The transportation problem; A streamlined simplex method for the transportation problem; The assignment problem; A special algorithm for the assignment problem.

7 Hours

UNIT - 7

GAME THEORY, DECISION ANALYSIS: Game Theory: The formulation of two persons, zero sum games; Solving simple games- a prototype example; Games with mixed strategies; Graphical solution procedure; Solving by linear programming; Extensions.
Decision Analysis: A prototype example; Decision making without experimentation; Decision making with experimentation; Decision trees.

6 Hours

UNIT - 8

METAHEURISTICS: The nature of Metaheuristics, Tabu Search, Simulated Annealing, Genetic Algorithms.

6 Hours

TEXT BOOK:

1. **Introduction to Operations Research** - Frederick S. Hillier and Gerald J. Lieberman, 8th Edition, Tata McGraw Hill, 2005.

REFERENCE BOOKS:

1. **Operations Research Applications and Algorithms** - Wayne L. Winston, 4th Edition, Thomson Course Technology, 2003.
2. **Operations Research: An Introduction** - Hamdy A Taha, 8th Edition, Prentice Hall India, 2007.